



Extension of Picard's Successful Iteration Method to Numerical Solution of Second Order Initial Value Problem

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ABSTRACT

In this paper, Picard's successive iteration method, to obtain the numerical solution for second order initial value problem is derived. The derivation of the new method is purely based on differentiation of the general class of the problem, following the Picard's first order iteration method steps with little modification and adjustment. The derived method is implemented on three numerical examples, and the result obtained converge positively to the exact solution as contained in literature.

Keywords: Picard's successive method, Initial Value Problem, Ordinary differential equation

INTRODUCTION

In the field of engineering and science, we come across physical and natural phenomenon which, when represented by mathematical models, happen to be differential equation of motion, equation of motion, deflection of a beam etc., are represented by differential equations. Hence the solution of differential equation is a necessity in such studies. There are number of differential equations which we studied in Calculus to get closed form solutions. But, all differential equation do not possess closed form or finite form solution. Even if they possess closed form solutions, we do not know the method of getting it. In such situations, depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the numerical solutions of the differential equations. No doubt, such numerical solutions are approximate solutions. But, in many cases, approximate solutions to the required accuracy are quite sufficient, (Kandasamy et al. 2008).

The successive approximations method, also called the Picard iteration method provides a scheme that can be used for solving initial valued problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in recurrence relation

to determine the other approximation, (Wazwaz, 2011).

In this paper we consider second order initial value problem of the form

$$y'' = f(x, y, y') \quad (1.1)$$

$$y(x_0) = y_0, y'(x_0) = y'_0 \quad (1.2)$$

Equation (1.1) together with equation (1.2) has already been solved directly by several researchers such as (Chakravarti, and Worland, 1971), (Suleiman, 1989), (Fatunla, 1991) and (Omar and Suleiman, 2005). The system of higher order ODEs can be reduced to a system of first order equation and then solved using first order ODEs. This approach will enlarge the system of first order ODEs and needs more computational work. The main idea of this research is to extend the Picard's iterative method for solving equation (1.1) together with (1.2) directly without reducing it to first order type. Numerical results are given to show the efficiency of the proposed method.

Definition of relevant terms and theorems

Differential Equations

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation, (Adeniyi, 2007).

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Examples of differential equations is

$$\frac{d^2y}{dx^2} + xy \left(\frac{dy}{dx}\right)^2 = 0 \tag{2.1}$$

Order of a Differential Equations

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation, (Adeniyi, 2007). The ordinary differential equation (2.1) is of the second order, since the highest derivative involved is a second derivatives.

Ordinary Differential Equations

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation, (Adeniyi, 2007). Example of an ordinary differential equation is

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{2.3}$$

Partial Differential Equations

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation, (Adeniyi, 2007). Example of a partial differential equation is

$$\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} + \frac{\partial^2u}{\partial z^2} = 0 \tag{2.4}$$

Linear Ordinary Differential Equations

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_0(x)y = b(x) \tag{2.5}$$

Where a_0 is not identically zero.

Example of a linear ordinary differential equation is

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \tag{2.52}$$

Non-Linear Ordinary Differential Equation

A non-linear ordinary differential equation is an ordinary differential equation that is not linear. Example of a non-linear differential equation is

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \tag{2.6}$$

Theorem 1: The Existence and Uniqueness Theorem

Consider the differential equation

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \tag{2.7}$$

Suppose x_0 is a given ‘‘initial point’’ $x = x_0$, and suppose a_0, a_1, \dots, a_{n-1} are given constants. Then there is exactly one solution to the differential equation (1) which satisfies the initial conditions.

$$y(x_0) = a_0, y'(x_0) = a_1, y''(x_0) = a_2, \dots, y^{(n-1)}(x_0) = a_{n-1}. \tag{2.8}$$

Note that for an n^{th} order equation we can prescribe exactly n initial values.

Reducing Multiple Integral to Single Integral

The general formula that converts multiple integrals to single integral is given by

$$\int_0^x \int_0^{x_1} \dots \int_0^{x_{j-1}} u(x_j) dx_j dx_{j-1} \dots dx_1 = \frac{1}{(j-1)!} \int_0^x (x-t)^{(j-1)} u(t) dt. \tag{2.9}$$

Equation (2.9) for $j=2$ we have

$$\int_{x_0}^x \int_{x_0}^x f(t, y(t), y'(t)) dt dt = \int_{x_0}^x (x-t) f\left(t, y(t) \frac{dy}{dt}\right) dt \tag{3.0}$$

This formula will be used to convert initial value problem to Volterra integral equation.

DERIVATION OF PICARD’S SUCCESSIVE ITERATION METHOD FOR SECOND ORDER INITIAL VALUE PROBLEM

The derivation of the Picard’s successive iteration method for the solution of second order Initial Value Problem as given in equation (1) is discussed in this section.

Consider

$$y'' = f(x, y, y'), \tag{3.1}$$

together with the condition

$$y(x_0) = y_0, y'(x_0) = y'_0 \tag{3.2}$$

Integrating (3.1) with respect to x we obtain

$$y'(x) = y'(0) + \int_{x_0}^x f(t, y, y') dt \tag{3.3}$$

Integrating (3.3) with respect to x

$$\int_{x_0}^x y'(t) dt = \int_{x_0}^x y'(0) dt + \int_{x_0}^x \int_{x_0}^x f(t, y(t), y'(t)) dt dt \tag{3.4}$$

Substituting (3.0) on (3.4) we have

$$y(x) - y_0 = (x - x_0)y'_0 + \int_{x_0}^x (x-t) f\left(t, y(t) \frac{dy}{dt}\right) dt \tag{3.6}$$

Thus,

$$y_{(m+1)} = y_o + (x - x_o)y'_o + \int_{x_o}^x (x - t)f(f(t, y_m(t), y'_m(t)))dt \quad (3.7)$$

Equation (3.7) is thus the developed Picard's iteration scheme needed to solve second order initial value problems.

Equation (3.7) for $x_0 = 0$ will give

$$y_{(m+1)} = y_o + y'_o + \int_{x_o}^x (x - t)f(f(t, y_m(t), y'_m(t)))dt \quad (3.8)$$

Numerical Examples

Here we considered four examples to illustrate the solvability using the newly developed method.

Example 1:

Consider second order differential equation

$$y'' - y' = 0 \quad (4.1)$$

Together with the initial conditions

$$y(0) = 2, \quad y'(0) = 1 \quad (4.2)$$

Exact solution is $y(x) = e^x + 1$

Solution:

Rewrite (4.1) in the form

$$y'' = y' \quad (4.3)$$

$$y_{m+1} = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_m(t), y'_m(t))dt \quad (4.4)$$

Making use of (4.4) on (4.3) we obtain

For m=0 we have

$$y_1 = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_0(t), y'_0(t))dt \quad (4.5)$$

Thus,

$$y_1 = 2 + x + \int_{x_o}^x (x - t)[1]dt \quad (4.6)$$

$$y_1 = 2 + x + x^2 - \frac{x^2}{2} \quad (4.7)$$

$$y_1 = 2 + x + \frac{x^2}{2} \quad (4.8)$$

Equation (4.8) can be expressed as

$$y_1 = 1 + 1 + x + \frac{x^2}{2} \quad (4.9)$$

For m =1 we have

$$y_2 = y_o + xy'_o + \int_{x_o}^x (x - t)f[t, y_1(t), y'_1(t)]dt \quad (4.10)$$

$$y_2 = 2 + x + \int_{x_o}^x (x - t)(1 + t)dt \quad (4.11)$$

$$y_2 = 2 + x + \int_{x_o}^x (x + xt - t - t^2)dt \quad (4.12)$$

$$y_2 = 2 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad (4.13)$$

Equation (4.13) can be expressed as

$$y_2 = 1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad (4.14)$$

For m=2

$$y_3 = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_2(t), y'_2(t))dt \quad (4.15)$$

Thus,

$$y_3 = 2 + x + \int_{x_o}^x (x - t) \left(1 + t + \frac{t^2}{2}\right) dt \quad (4.16)$$

$$y_3 = 2 + x + \int_{x_o}^x \left(x + xt + \frac{xt^2}{2} - t - t^2 - \frac{t^3}{2}\right) dt \quad (4.17)$$

$$y_3 = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \quad (4.18)$$

For m=4

$$y_4 = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \quad (4.19)$$

Subsequently,

$$y_5 = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \quad (4.20)$$

$$y_6 = 2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \quad (4.21)$$

Conclusively therefore, for m= k-1, we have

$$y_{k-1} = 1 + \sum_{m=0}^{k-1} \frac{x^m}{m!} \quad (4.22)$$

Which gives as $m \rightarrow \infty$

$$y = 1 + e^x \quad (4.23)$$

Which is the exact solution in closed form.

Example 2:

Consider the Second order differential equation

$$y'' + y = 0, \quad (4.24)$$

Together with the initial conditions

$$(0) = 1, \quad y'(0) = 0 \quad (4.25)$$

$$\text{Exact solution } y(x) = \cos x \quad (4.26)$$

$$y'' + y = 0 \Rightarrow y'' = -y \quad (4.27)$$

$$y_{m+1} = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_m(t), y'_m(t))dt \quad (4.28)$$

When m=0

$$y_1 = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_0(t), y'_0(t))dt \quad (4.29)$$

$$y_1 = y_o + xy'_o + \int_{x_o}^x (x - t)(-y_o)dt \quad (4.30)$$

$$y_1 = 1 + \int_{x_o}^x (x - t)(-1)dt \quad (4.31)$$

$$y_1 = 1 + \int_{x_o}^x (-x + t)dt \quad (4.32)$$

$$y_1 = 1 - x^2 + \frac{x^2}{2} \quad (4.33)$$

$$y_1 = 1 - \frac{x^2}{2} \quad (4.34)$$

When m=1

$$y_2 = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_1(t), y'_1(t))dt \quad (4.35)$$

$$y_2 = 1 - \int_{x_o}^x (x - t) \left(1 - \frac{t^2}{2}\right) dt \quad (4.36)$$

$$y_2 = 1 - \left[x^2 - \frac{x^4}{6} - \frac{x^2}{2} - \frac{x^4}{8}\right] \quad (4.37)$$

$$y_2 = 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad (4.38)$$

When m=2

$$y_3 = y_o + xy'_o + \int_{x_o}^x (x - t)f(t, y_2(t), y'_2(t))dt \quad (4.38)$$

$$y_3 = 1 - \int_{x_o}^x (x - t) \left(1 - \frac{t^2}{2} + \frac{t^4}{24}\right) dt \quad (4.39)$$

$$y_3 = 1 - \int_{x_o}^x (x - t) \left(1 - \frac{t^2}{2} + \frac{t^4}{24}\right) dt \quad (4.40)$$

$$y_3 = 1 - \int_0^x \left(x - \frac{xt^2}{2} + \frac{xt^4}{24} - t + \frac{t^3}{2} + \frac{t^5}{24}\right) dt \quad (4.41)$$

$$y_3 = 1 - \left[x^2 - \frac{x^4}{6} + \frac{x^6}{5!} - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^4}{6!} \right] \quad (4.42)$$

$$y_3 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \quad (4.43)$$

Subsequently,

$$y_4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \quad (4.44)$$

$$y_5 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \quad (4.45)$$

Conclusively therefore, for m= k-1, we have

$$y_{k-1} = \sum_{m=0}^{k-1} \frac{(-1)^m}{(2m)!} x^{2m} \quad (4.46)$$

Which gives as m → ∞

$$y = \cos x \quad (4.47)$$

Which is the exact solution.

Example 3:

Consider second Order Differential Equation

$$y'' = y + xy \quad (4.48)$$

Together with the initial conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (4.49)$$

Exact solution is

Solution:

$$y_{m+1} = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_m(t), y'_m(t))dt \quad (3.13)$$

Making use of (3) on (1) we obtain

For m=0 we have

$$y_1 = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_0(t), y'_0(t))dt \quad (4.50)$$

$$y_1 = y_0 + x(0) + \int_{x_0}^x (x-t)1dt \quad (4.51)$$

$$y_1 = 1 + x^2 - \frac{x^2}{2} \quad (4.52)$$

$$y_1 = 1 + \frac{x^2}{2} \quad (4.53)$$

When m=1 we have

$$y_2 = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_1(t), y'_1(t))dt \quad (4.54)$$

$$y_2 = 1 + \int_{x_0}^x (x-t) \left[t^2 - t \left(1 - \frac{3t^2}{2} \right) \right] dt \quad (4.55)$$

$$y_2 = 1 + \frac{x^2}{2} + \frac{x^4}{4} \quad (4.56)$$

When m=2

$$y_3 = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_2(t), y'_2(t))dt \quad (4.57)$$

$$y_3 = 1 - \int_{x_0}^x (x-t) \left(1 + \frac{t^2}{2} + \frac{t^4}{24} + t[t + t^3] \right) dt \quad (4.58)$$

$$y_3 = 1 - \int_{x_0}^x (x-t) \left(1 - \frac{3t^2}{2} + \frac{5t^4}{4} \right) dt \quad (4.59)$$

$$y_3 = 1 + \frac{x^2}{2!} + \frac{x^4}{8} + \frac{x^6}{24} \quad (4.60)$$

Which give same result as the power series method[1].

Example 4:

Consider second Order Differential Equation

$$y'' = x^2 - xy \quad (4.61)$$

Together with the initial conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (4.62)$$

Exact solution is

Solution:

$$y_{m+1} = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_m(t), y'_m(t))dt \quad (3.13)$$

Making use of (3) on (1) we obtain

For n=0 we have

$$y_1 = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_0(t), y'_0(t))dt \quad (4.63)$$

$$y_1 = y_0 + x(0) + \int_{x_0}^x (x-t)[t^2 - t(1)]dt \quad (4.64)$$

$$y_1 = 1 + \int_{x_0}^x (x-t)(t^2 - t)dt \quad (4.65)$$

$$y_1 = 1 + \int_{x_0}^x (xt^2 - xt - t^3 + t^2) dt \quad (4.66)$$

$$y_1 = 1 + \frac{x^4}{3} - \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^3}{3} \quad (4.67)$$

$$y_1 = 1 - \frac{x^3}{6} + \frac{x^4}{12} \quad (4.68)$$

When n=1 we have

$$y_2 = y_0 + xy'_0 + \int_{x_0}^x (x-t)f(t, y_1(t), y'_1(t))dt \quad (4.67)$$

$$y_2 = 1 + \int_{x_0}^x (x-t) \left[t^2 - t \left(1 - \frac{t^3}{6} + \frac{t^4}{12} \right) \right] dt \quad (4.68)$$

$$y_2 = 1 + \frac{x^4}{3} - \frac{x^3}{2} - \frac{x^6}{30} + \frac{x^7}{72} - \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^6}{36} - \frac{x^7}{72} \quad (4.69)$$

$$y_2 = 1 - \frac{x^3}{6} + \frac{x^4}{12}$$

$$y_3 = 1 - \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^6}{180}$$

Conclusion

In this article, we have investigated the solvability of second order initial value problem with Picard's iteration method. The background and the numerical procedures of the existing Picard's method for first order initial value problems were strictly followed with little modification where necessary. We have chosen three testproblems to evaluate the effectiveness of the developed Picard's iteration method and compare the convergence with existing rational methods in terms of numerical accuracy. Numerical experimentations and comparisons seemed to indicate that Picard's iteration methods were more accurate.

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