



The Differential Variant and Its Error Estimate for the Class of Non-Overdetermined Third Order Ordinary Differential Equation

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ABSTRACT

The tau method for the numerical solution of Ordinary Differential Equations (ODEs) seeks an approximant of the solution which exactly satisfies the corresponding Perturbed Ordinary Differential Equations (PODE). This tau approximant has the power of x as its basis function. In this paper, the differential variant of the tau method for initial value problems (IVPs) in the class of non-overdetermined third order ordinary differential equations is considered. The corresponding error estimate for the variant is obtained and numerical results for some selected problems were provided. The order of the approximant is closely captured based on the numerical evidences obtained.

Keywords: Approximant, Error Estimate, Non-over determined, Tau Method, Variant

INTRODUCTION

The tau method was originally developed by Lanczos (Lanczos, 1938; Lanczos, 1956) for the solution class of $m - th$ order problem

$$Ly(x) := \sum_{r=0}^m P_r(x)y^{(r)}(x) = f(x) \quad (1.1a)$$

$$Ly(x) := \sum_{r=0}^m a_{rk}(x)y^{(r)}(x_{rk}) = \alpha_k, k = 1(1)m \quad (1.1b)$$

where $y^{(r)}$ stands for the derivatives of order $y(x), f(x)$ and $p(x), r = 0(1)n$ are polynomials

(or polynomial approximations immediately derivable by using the tau method) of given functions, where a_{rk}, x_{rk} and α_k are given real numbers. The method solves problem (i) by seeking an approximation

$$y_n(x) := \sum_{r=0}^n a_r x^{(r)}, n < \infty \quad (1.2)$$

which is the exact solution of the perturbed form

$$Ly_n(x) := P_r(x)y^{(x)}(x) = f(x) + H_n(x) \quad (1.3)$$

which is the conditions stated in (1.1b)

$H_n(x)$ is the linear combination of Chebyshev polynomials (Fox and Parker, 1968; Ortiz, 1974) valid in the interval $[a, b]$ and it may be of the form

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n-m+r+1}(x) \quad (1.4)$$

The parameter S in (1.4) is the number of over determination of (1.3), τ_{m+s+1} are the free tau parameters to be determined by inserting (1.2) and (1.3) and then applying the conditions

(1.1b), we get the system of linear equations in $(n + m + s + 1)$ unknown constants, $a_r, r = 0(1)n$

and $\tau_1, \tau_2 \dots \tau_{m+s}$. The system of linear equation in $(n + m + s + 1)$ are then solved to obtain the $(n + m + s + 1)$ unknown constants which are substituted into (1.1b) to get our approximate solution of (1.1a)

Differential Form of the Tau Method

A Generalized Formulation for the Integrated Variant of Tau Method for Over determined $m - th$ order ordinary differential equations was reported (Badeggi *et al.*, 2017) and Ma'ali *et al.* (2016) reported derivation of three variants of τ -methods for the Class of Non-overdetermined Ordinary Differential Equations. Also the differential formulation of the Tau method and its error estimate for fourth order non-overdetermined differential equations was reported and (Ma'ali *et al.*, 2016).

Thus, in this paper, we shall be considering the class of non-overdetermined third order linear ordinary differential equations, that is,

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$$(P_{30} + P_{31}x + P_{32}x^2 + P_{33}x^3) y'''(x) + (P_{20} + P_{21}x + P_{22}x^2) y''(x) + (P_{10} + P_{11}x) y'(x) + P_{00}y(x) = f(x), \quad a \leq x \leq b \quad (1.5a)$$

Where P_{rk} , $r, k = 0(1)3$ are constants and $f(x)$ is a polynomial, together with the associated conditions

$$y(a) = \alpha_1, y'(a) = \alpha_2, y''(a) = \alpha_3 \quad (1.5b)$$

The differential form of the tau method which is one of the variant of this method with its corresponding error estimation technique will be applied to the class of IVPs (1.5)

Definition 1.1

The differential system

$$Ly_n(x) \equiv \sum_{r=0}^m P_r(x) y_n^{(r)}(x) = f(x) + H_n(x) \quad (1.6a)$$

$$Ly_n(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y_n^{(r)}(x_{rk}) = \alpha_k, k = 1(1)m \quad (1.6b)$$

will be called the Tau problem corresponding to the differential system

$$Ly(x) \equiv \sum_{r=0}^m P_r(x) y^{(r)}(x) = f(x) \quad (1.7a)$$

$$Ly(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, k = 1(1)m \quad (1.7b)$$

We call the n th degree polynomial $y_n(x)$ which satisfies the tau problem (1.6), the Tau approximant of (1.7) and the tau solution of (1.6)

Definition 1.2

The number of over-determinations, S of (1.1a) is defined as

$$S = \text{Max} \{N_r - r; 0 \leq r \leq m\} \quad (1.8)$$

for $N_r \geq r$ and $0 \leq r \leq m$

Definition 1.3

The equation (1.1a) is said to be non-overdetermined if S given by (1.8) is zero i.e if $S=0$, otherwise is over-determined

Definition 1.4

The system of equation $A \underline{\tau} = \underline{\beta}$ where $\underline{\tau} = (a_0, a_1, a_2, \dots, a_n, \tau_1, \tau_2, \tau_3 \dots \tau_{m+s})^T$ resulting the process of solution of (1.7) will be referred to as Tau system of (1.6)

Error Estimation of the Tau Method

Here, we shall review the error estimation of the tau method for the differential formulation of the preceding section and which had been reported by Ma'ali et al. (2013;2014)

Error Estimation for the Differential Formulation.

The error of $e_n(x)$ of the tau approximant $y_n(x)$ of (1.1) given by

$$e_n(x) = y(x) - y_n(x), a \leq x \leq b \quad (2.1)$$

Satisfies the equation

$$\sum_{r=0}^n P_r(x) y^{(r)}(x) = -H_n(x) \quad (2.2)$$

Let us consider the perturbed error equation

$$L(e_n(x))_{n+1} = -H_n(x) + H_m(x) \quad m \geq n + 1 \quad (2.3)$$

where,

$$H_m(x) = \sum_{r=0}^{m+s-1} \hat{\tau}_{m+s-r} T_{-m+r+2}(x) \quad (2.4a) - H_n(x) \equiv \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \quad (2.4b)$$

$$Le_n(x_{rk}) = 0, \quad k = 1(1)m \quad (2.4c)$$

The necessary $(m + s - 1)$ equations to determine $\hat{\tau}_r, \dots, \hat{\tau}_{m+s}$ and φ_n are given by equating corresponding coefficient of $x^{n+s-1}, \dots, x^{n-m+1}$ in equation (2.3).

A forward elimination is used Substituting the expression so obtained for φ_n into (2.5) gives $(e_n(x))_{n+1}$ in terms of known quantities

$$(e_n(x))_{n+1} = \frac{\mu_m(x) \varphi_n \tau_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \quad (2.5)$$

Thus, we obtain, for large n , the estimate

$$\varepsilon = \text{Max}_{a \leq x \leq b} |(e_n(x))_{n+1}| = \left| \frac{\varphi_n}{C_{n-m+1}^{(n-m+1)}} \right| = \text{Max}_{a \leq x \leq b} |e_n(x)| \quad (2.6)$$

Tau Approximant by the Differential Form

We consider here, derivation of Tau approximant and their error estimates for the class of non-overdetermined third order ordinary differential equations.

$$Ly(x) = (P_{30} + P_{31}x + P_{32}x^2 + P_{33}x^3) y'''(x) + (P_{20} + P_{21}x + P_{22}x^2) y''(x) + (P_{10} + P_{11}x) y'(x) + P_{00}y(x) = \sum_{r=0}^n f_r x^r, \quad a \leq x \leq b \quad (3.1a)$$

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2 \quad (3.1b)$$

i. e In this case $m = 3, s = 0$

Since the transformation

$$u = \frac{(x - a)}{(b - a)}, a \leq x \leq b$$

From (1.2) and (1.8) for $m = 3$ and $s = 0$, that is, for corresponding to (3.1) we have

$$(P_{30} + P_{31}x + P_{32}x^2 + P_{33}x^3) \sum_{r=0}^n r(r-1)(r-2)a_r x^{r-3} + (P_{20} + P_{21}x + P_{22}x^2) \sum_{r=0}^n r(r-1)a_r x^{r-2} + (P_{10} + P_{11}x) \sum_{r=0}^n r a_r x^{r-1} + P_{00} \sum_{r=0}^n r a_r x^r = \sum_{r=0}^n f_r x^r + \tau_1 T_n(x) + \tau_2 \tau_{n-1}(x) + \tau_3 \tau_{n-2}(x) \tag{3.2}$$

where ,

$$T_n(x) = \sum_{r=0}^n C_r^{(n)} x^r, T_{n-1}(x) = \sum_{r=0}^{n-1} C_r^{(n-1)} x^r \text{ and } T_{n-2}(x) = \sum_{r=0}^{n-2} C_r^{(n-2)} x^r$$

which leads to,

$$\sum_{k=0}^n [P_{33}k(k-1)(k-2) + P_{22}k(k-1) + P_{11}k + P_{00}]a_k x^k + \sum_{k=0}^{n-1} [P_{32}k(k+1)k(k-1) + P_{21}k(k-1)k + P_{10}(k+1)]a_{k+1} x^{k+1} + \sum_{k=0}^{n-2} [P_{31}k(k+1)(k+2) + P_{20}k(k+2)(k+1)]a_{k+2} x^k + \sum_{k=0}^{n-3} [P_{30}(k+3)(k+2)(k+1)]a_{k+3} x^k = \sum_{k=0}^n f_r x^r + \tau_1 \sum_{k=0}^n C_k^{(n)} x^k + \tau_2 \sum_{k=0}^{n-1} C_k^{(n-1)} x^k + \tau_3 \sum_{k=0}^{n-2} C_k^{(n-2)} x^k \tag{3.3}$$

Thus, we have

$$\begin{aligned} & \left[\{n(n-1)(n-2)P_{33} + n(n-1)P_{22} + nP_{11} + P_{00}\}a_n - f_n - \tau_1 C_n^{(n)} \right] x^n + \left[\{(n-1)(n-2)(n-3)P_{33} + n(n-1)(n-2)P_{22} + (n-1)P_{11} + P_{00}\}a_{n-1} - f_{n-1} - \tau_1 C_{n-1}^{(n)} + \{n(n-1)(n-2)P_{32} + n(n-1)P_{21} + nP_{10}\}a_n - \tau_2 C_{n-1}^{(n)} \right] x^{n-1} \\ & + \left[\{n(n-2)(n-3)P_{33} + n(n-2)(n-3)P_{22} + (n-2)P_{11} + P_{00}\}a_{n-2} - f_{n-2} - \tau_1 C_{n-2}^{(n)} \right] x^{n-2} \\ & + \{n(n-2)(n-3)(n-4)P_{33} + n(n-2)(n-3)P_{22} + (n-2)P_{11} + P_{00}\}a_{n-2} - f_{n-2} - \tau_1 C_{n-2}^{(n)} \\ & + \{n(n-1)(n-2)P_{31} + n(n-1)P_{20}\}a_n - \tau_2 C_{n-2}^{(n-1)} - \tau_3 C_{n-2}^{(n-2)} \Big] x^{n-2} + n(n+1)(n-1)a_{n+1} P_{30} = 0 \end{aligned} \tag{3.4}$$

By equating corresponding coefficient of the linear system we have

$$\{k(k-1)(k-2)P_{33} + k(k-1)P_{22} + kP_{11} + P_{00}\}a_k + \{k(k+1)k(k-1)P_{32} + k(k+1)kP_{21} + (k+1)P_{10}\}a_{k+1} + \{k(k+1)(k+2)kP_{31} + k(k+1)(k+2)P_{20}\}a_{k+2} + (k+1)(k+2)(k+3)a_{k+3}P_{30} -$$

$$\tau_1 C_k^{(n)} - \tau_2 C_k^{(n-1)} - \tau_3 C_k^{(n-2)} - f_k = 0, k = 0(1)n-3,$$

which yields

$$\begin{aligned} & \{(n-2)(n-3)(n-4)P_{33} + (n-2)(n-3)P_{22} + (n-2)P_{11} + P_{00}\}a_{n-2} + \\ & \{(n-1)(n-2)(n-3)P_{32} + (n-1)(n-2)P_{21} + (n-1)P_{10}\}a_{n-2} + \{n(n-1)(n-2)P_{31} + n(n-1)P_{20}\}a_n - \tau_1 C_{n-2}^{(n)} - \\ & \tau_2 C_{n-2}^{(n-1)} - \tau_3 C_{n-2}^{(n-2)} - f_n = 0 \end{aligned} \tag{3.5a}$$

$$\{(n-1)(n-2)(n-3)P_{33} + (n-1)(n-2)P_{22} + (n-1)P_{11} + P_{00}\}a_{n-1} + \{n(n-1)(n-2)P_{32} + n(n-1)P_{21} + nP_{10}\}a_n - \tau_1 C_{n-2}^{(n)} - \tau_2 C_{n-2}^{(n-1)} - f_{n-1} = 0 \tag{3.5b}$$

$$\{n(n-1)(n-2)P_{33} + n(n-1)P_{22} + nP_{11} + P_{00}\}a_n - \tau_1 C_n^{(n)} - f_n = 0 \tag{3.5c}$$

The solutions of this system together with the two equations arising from condition (3.1b) for $a_r, r = 0(1)n$ and τ_1, τ_2, τ_3 which is subsequently leads to the approximanty $_n(x)$

Error Estimation for Differential Form

Given that,

$$\begin{aligned} Ly_n(x) & \equiv (P_{30} + P_{31}x + P_{32}x^2 + P_{33}x^3) y'''(x) + (P_{20} + P_{21}x + P_{22}x^2) y''(x) \\ & + (P_{10} + P_{11}x) y'(x) + P_{00} y(x) = \sum_{r=0}^n f_r x^r, a \leq x \leq b \\ y(a) & = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2 \end{aligned} \tag{3.6}$$

Thus, for the problem above we have from (2.3)

$$L(e_n(x))_{n+1} = \hat{\tau}_1 T_{n+1}(x) + \hat{\tau}_2 T_n(x) + \hat{\tau}_3 T_{n-1}(x) - \tau_1 T_n(x) - \tau_2 T_{n-1}(x) - \tau_3 T_{n-2}(x) \tag{3.6a}$$

$$L(e_n(x))_{n+1} = \hat{\tau}_1 T_{n+1}(x) + (\hat{\tau}_2 - \tau_1) T_n(x) + (\hat{\tau}_3 - \tau_2) T_{n-1}(x) - \tau_3 T_{n-2}(x) \tag{3.6b}$$

$$(e_n(0))_{n+1} = 0, (e'_n(0))_{n+1} = 0 \tag{3.6c}$$

where,

$$(e_n(x))_{n+1} = \frac{x^3 \phi_n T_{n-2}(x)}{C_{n-2}^{(n-2)}} = \frac{\phi_n \sum_{r=0}^{n-2} C_{n-2}^{(n-2)} x^{r+3}}{C_{n-2}^{(n-2)}} \tag{3.7}$$

From the coefficient of x^{n+1}, x^n, x^{n-1} and x^{n-2} we obtain the following system of equations

$$\begin{aligned} &\frac{\emptyset_n}{C_{n-2}^{(n-2)}} \{P_{33}(n-1)(n+1)C_{n-2}^{(n-2)} + P_{22}n(n+1)C_{n-2}^{(n-2)} + P_{11}(n+1)C_{n-2}^{(n-2)} + P_{00}C_{n-2}^{(n-2)}\} = \hat{\tau}_1\{C_{n+1}^{(n+1)}\} \\ &\frac{\emptyset_n}{C_{n-2}^{(n-2)}} \{P_{32}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{33}n(n-1)(n-2)C_{n-3}^{(n-2)} + P_{21}n(n+1)C_{n-2}^{(n-2)}\} \\ &+ P_{22}n(n-1)C_{n-3}^{(n-2)} + P_{10}n(n+1)C_{n-2}^{(n-2)} + P_{11}nC_{n-3}^{(n-2)} + P_{00}C_{n-3}^{(n-2)}\} = \hat{\tau}_1\{C_n^{(n+1)}\} + (\hat{\tau}_2 - \tau_1)\{C_n^{(n)}\} \\ &\frac{\emptyset_n}{C_{n-2}^{(n-2)}} \{P_{31}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{32}n(n-1)(n-2)C_{n-3}^{(n-2)} + P_{33}(n-1)(n-2)(n-3)C_{n-4}^{(n-2)} + \\ &P_{20}n(n+1)C_{n-2}^{(n-2)} + P_{21}n(n-1)C_{n-3}^{(n-2)} + P_{22}(n-1)(n-2)C_{n-4}^{(n-2)} + nP_{10}C_{n-3}^{(n-2)} + P_{11}(n-1)C_{n-4}^{(n-2)} + P_{00}C_{n-4}^{(n-2)}\} \\ &= \hat{\tau}_1\{C_{n-1}^{(n+1)}\} + (\hat{\tau}_2 - \tau_1)C_{n-1}^{(n)} + (\hat{\tau}_3 - \tau_2)C_{n-1}^{(n-1)}\} \\ &\frac{\emptyset_n}{C_{n-2}^{(n-2)}} \{P_{30}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{31}n(n-1)(n-2)C_{n-3}^{(n-2)} + \\ &P_{32}(n-1)(n-2)(n-3)C_{n-4}^{(n-2)} + P_{33}(n-2)(n-3)(n-4)C_{n-5}^{(n-2)} + \\ &P_{20}n(n-1)C_{n-3}^{(n-2)} + P_{21}(n-1)(n-2)C_{n-4}^{(n-2)} + P_{22}(n-2)(n-3)C_{n-5}^{(n-2)} + \\ &P_{10}(n-1)C_{n-4}^{(n-2)} + P_{11}(n-2)C_{n-5}^{(n-2)} + P_{00}C_{n-5}^{(n-2)}\} = \hat{\tau}_1\{C_{n-2}^{(n+1)}\} + (\hat{\tau}_2 - \tau_1)C_{n-2}^{(n)} \\ &+ (\hat{\tau}_3 - \tau_2)\{C_{n-2}^{(n-1)} - \tau_3C_{n-2}^{(n-2)}\} \end{aligned} \tag{3.8}$$

which yields,

$$\begin{aligned} &\emptyset \left[P_{33}(n-1)n(n+1)C_{n-2}^{(n-2)} + P_{22}(n+1)nC_{n-2}^{(n-2)} + P_{11}(n+1)C_{n-2}^{(n-2)} + P_{00}C_{n-2}^{(n-2)} \right] = \hat{\tau}_1 C_{n+1}^{(n+1)} \\ &\emptyset [P_{32}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{33}n(n-1)(n-2)C_{n-3}^{(n-2)} + P_{21}n(n+1)C_{n-2}^{(n-2)} \\ &+ P_{22}n(n-1)C_{n-3}^{(n-2)} + P_{10}(n+1)C_{n-2}^{(n-2)} + P_{11}nC_{n-3}^{(n-2)} + P_{00}nC_{n-3}^{(n-2)}] = \hat{\tau}_1 C_n^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_n^{(n)} \\ &\emptyset [P_{31}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{32}n(n-1)(n-2)C_{n-3}^{(n-2)} \\ &+ P_{33}(n-1)(n-2)(n-3)C_{n-4}^{(n-2)} + P_{20}n(n+1)C_{n-2}^{(n-2)} + P_{21}n(n-1)C_{n-3}^{(n-2)} \\ &+ P_{22}(n-1)(n-2)C_{n-4}^{(n-2)} + P_{10}nC_{n-3}^{(n-2)} + P_{11}(n-1)C_{n-4}^{(n-2)} + P_{00}C_{n-4}^{(n-2)}] = \hat{\tau}_1 C_{n-1}^{(n+1)} + (\hat{\tau}_2 - \tau_1)C_{n-1}^{(n)} + (\hat{\tau}_3 - \tau_2)C_{n-1}^{(n-1)} \\ &\emptyset [P_{30}(n+1)n(n-1)C_{n-2}^{(n-2)} + P_{31}n(n-1)(n-2)C_{n-3}^{(n-2)} + \\ &P_{32}(n-1)(n-2)(n-3)C_{n-4}^{(n-2)} + P_{33}(n-2)(n-3)(n-4)C_{n-5}^{(n-2)} + \\ &P_{20}n(n-1)C_{n-3}^{(n-2)} + P_{21}(n-1)(n-2)C_{n-4}^{(n-2)} + P_{22}(n-2)(n-3)C_{n-5}^{(n-2)} + \\ &P_{10}(n-1)C_{n-4}^{(n-2)} + P_{11}(n-2)C_{n-5}^{(n-2)} + P_{00}C_{n-5}^{(n-2)}] = \hat{\tau}_1\{C_{n-2}^{(n+1)}\} + (\hat{\tau}_2 - \tau_1)C_{n-2}^{(n)} + \\ &(\hat{\tau}_3 - \tau_2)\{C_{n-2}^{(n-1)} - \tau_3C_{n-2}^{(n-2)}\} \end{aligned} \tag{3.9}$$

where,

$$\emptyset = \frac{\emptyset_n}{C_{n-2}^{(n-2)}}$$

Using well-known relations,

$$C_n^{(n)} = 2^{2n-1}, C_{n-1}^{(n)} = -\frac{1}{2}nC_n^{(n)}, C_{n-1}^{(n-1)} = -n2^{2n-1} \tag{3.10}$$

Solving (3.9) by forward elimination for \emptyset_n we obtain

$$\emptyset_n = \frac{2^{2n-5} \tau_3}{k} \tag{3.11}$$

$$\begin{aligned} &K = \{[(n+1)(n-1)P_{30} - (n-1)(n-3)P_{32} + (n-1)(n-2)P_{21} + (n-1)V]C_{n-4}^{(n-2)} \\ &+ [(n-2)(n-3)(n-4)P_{33} + (n-2)P_{11} + P_{00}]C_{n-5}^{(n-2)} - n(n-1)(n-2)^2P_{31} + n(n-1)(n-2)P_{20}\} \end{aligned} \tag{3.12}$$

Thus, we have the error estimate from (2.6)

$$\varepsilon = \frac{2^{2n-10}|\tau_3|}{k}$$

Numerical Examples

We consider the application of our error estimation formula obtained in the preceding section for somethree (3) selected problems for experimentation.

The exact error is defined as

$$\varepsilon^* = \max_{a \leq x \leq b} \{|y(x_k) - y_n(x_k)|\}$$

where ,

$$\{x_k\} = \{0.01k\}, \quad \text{for } k = 0(1)100$$

The numerical results were presented in the tables below.

Problem I

Third Order Non-Homogenous Problem

$$y'''(x) + 8y' = -6y' + 9x + 2, y(0) = 0, y'(0) = 0, y''(0) = \frac{11}{128}$$

with the exact solution

$$y(x) = \frac{11x^2}{256} + \frac{7x^3}{32} - \frac{x^4}{16}, 0 \leq x \leq 1$$

The numerical results are represented in table I below

Table I: Error and Error Estimate for Problem I

N	2	3	4	5
Error				
Approximate	4.21×10^{-6}	3.65×10^{-7}	5.16×10^{-9}	9.13×10^{-10}
Error ε				
Exact Error ε^*	1.23×10^{-6}	2.11×10^{-7}	4.76×10^{-8}	3.63×10^{-10}

Problem II

$$y''' - y' + 2y = 2, y(0) = 0, y'(0) = 1, y''(0) = 0$$

The exact solution is

$$y(x) = \frac{1}{2}(5e^x - 4e^{-x} - e^{4x}), 0 \leq x \leq 1$$

The numerical results are represented in table II below

Table II: Error and Error Estimate for Problem II

N	2	3	4	5
Error				
Approximate	2.01×10^{-3}	4.27×10^{-5}	8.56×10^{-7}	7.92×10^{-9}
Error ε				
Exact ε^*	3.53×10^{-3}	1.41×10^{-5}	3.10×10^{-7}	4.05×10^{-9}

Problem III

$$y''' + y' = 0, y(0) = 0, y'(0) = 1, y''(0) = 2$$

The exact solution is

$$y(x) = 2(1 - \cos x) + \sin x, \quad 0 \leq x \leq 1$$

The numerical results are represented in table III below

Table III: Error and Error Estimate for Problem III

N	2	3	4	5
Error				
Approximate	6.31×10^{-2}	7.65×10^{-4}	4.84×10^{-6}	2.32×10^{-7}
Error ε				
Exact ε^*	1.94×10^{-2}	3.08×10^{-4}	2.32×10^{-6}	1.34×10^{-7}

Conclusion

The differential variant of the tau method for solution of initial value problems (IVPS) in a class of third order ordinary differential equations with non-overdetermination has been presented .The tau approximant of the desired analytic solution and the associated error estimate was obtained.For all the numerical examples considered the error estimate closely approximate the exact error. The effectiveness of the method was demonstrated as the order of the tau approximant was estimated.

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