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RESEARCH PAPER

Trigonometrically Fitted Three Stage Third Order Improved Runge-Kutta Method for Numerical Integration of IVPs with Oscillating Solutions

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ABSTRACT

In this paper, a new trigonometrically fitted Improved Runge-Kutta (TFIRK) method is constructed and implemented. The new method is based on an explicit three stage Improved Runge-Kutta (IRK) method of order three. The convergence of the method is established and numerical results from its application to oscillatory problems show the accuracy and computational efficiency of the scheme.

Keywords: Trigonometrically-fitting, Initial Value Problems (IVPs), Improved Runge-Kutta method, oscillating solutions

INTRODUCTION

In the last few decades, there has been a growing interest in the research of new numerical techniques for approximating the solution of first order initial value problem of the form.

$$y'(x) = f\big(x, y(x)\big), y(x_0) = y_0 \quad x \epsilon [x_0, X] \qquad (1)$$

with periodic or oscillatory solutions. This type of problem arises in different fields of science and engineering, which includes quantum mechanics, classical mechanics, celestial mechanics, astrophysics, theoretical physics and chemistry, nuclear physics and biological sciences. In the quest for methods that best approximate the solution of (1), many authors considered different modifications to Runge-Kutta methods, such work can be seen in Geoken *et al.* (2000); Phohomsiri *et al.* (2004), Udwadia *et al.* (2008) and Xinyuan (2003). However most of the methods presented are obtained for the autonomous system while the Improved Runge-Kutta methods (IRK) can be used for autonomous as well as non-autonomous systems. Rabiei *et al.* (2011) constructed the new Improved Runge-Kutta method with reduced number of function evaluations. The method proposed is of order three with two stages. In furtherance to this, Rabiei *et al.* (2011) developed the fifth-order Improved Runge-Kutta method for solving ordinary differential equations. The method is of five stages.

In this paper we present a trigonometrically fitted method based on two-step explicit third and fourth order Improved Runge-Kutta method derived by Rabiei *et al.* (2013).

METHODOLOGY

Trigonometrically Fitted Methods

The Improved Runge-Kutta (IRK) method can be written as follows:

$$\begin{aligned} y_{n+1} = y_n + hb_1 f(x_n, y_n) - hb_{-1} f(x_{n-1}, y_{n-1}) \\ + h \sum_{i=2}^{s} b_i \big(f(x_n + c_i h, Y_i) \\ - f(x_{n-1} + c_i h, Y_{-i})\big) \qquad (2) \end{aligned}$$

$$Y_i = y_n + h \sum_{j=1}^{i-1} a_{ij} f\big(x_n + c_j h, Y_j\big) \qquad (3)$$

$$Y_{-i} = y_{n-1} + h \sum_{j=1}^{i-1} a_{ij} f\big(x_{n-1} + c_j h, Y_{-j}\big) \qquad (4)$$

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where y_{n+1} and y_n are approximations to $y(x_{n+1})$ and $y(x_n)$ respectively.
Trigonometrically-fitted methods are generally derived to exactly approximate the solution of the IVPs whose solutions are linear combination of the functions $\{_{\chi}{}^{j}e^{\alpha x}, _{\chi}{}^{j}e^{-\alpha x}\}$, where α can be complex or a real number.
Suppose $\mathrm{G}(x) = e^{\pm\alpha x}$, where $i = \sqrt{-1}$ is an imaginary unit, is the solution of (1). Applying $(2) - (4)$ to $\mathrm{G}(x)$ generates the Recursive relations (5) – (8) (Jikantoro *et al*., 2015).

$$cos(c_i z) = 1 - z\sum_{j=1}^{i-1} a_{ij}\sin(c_j z),\ \ i = 2,3,\dots,s \quad (5)$$

$$sin(c_i z) = z\sum_{j=1}^{i-1} a_{ij}\, cos(c_j z), \quad i = 2,3,\dots,s \quad (6)$$

$$cos(z) = 1 - zb_{-1}\, sin(z) - z\sum_{i=2}^{s} b_i\, sin(c_i z)$$

$$+z\sum_{i=2}^{s} b_i\, sin\big(z(c_i - 1)\big) \quad (7)$$

$$sin(z) = zb_1 - zb_{-1}\, cos(z) + z\sum_{i=2}^{s} b_i\, cos(c_i z)$$

$$-z\sum_{i=2}^{s} b_i\, cos\big(z(c_i - 1)\big) \quad (8)$$

The relations $(5) - (8)$ are the relations of order conditions of the trigonometrically fitted method. These relations replace the equations of order conditions of two-step Improved Runge-Kutta (IRK) method, which can be solved to give the coefficients of a particular method based on existing coefficients for solving problem of the form (1).
The order conditions up to order three of two-step IRK methods derived by Rabiei *et al*. (2013) is:

$$\left.\begin{array}{l} \text{order 1: } b_1 - b_{-1} = 1 \\ \text{order 2: } b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} \\ \text{order 3: } \sum_{i=2}^{s} b_i\, c_i = \frac{5}{12} \end{array}\right\} \quad (9)$$

All subscripts run to s or less.

Derivation of TFIRK Method with $s = 3, p = 3$

The Butcher array for the IRK3-3 method as derived by Rabiei *et al.*(2013) is

Table I Coefficients of IRK3-3

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -\frac{1}{3} & \frac{4}{3} & \\ \hline -\frac{1}{12} & \frac{11}{12} & \frac{1}{3} & \frac{1}{4} \end{array}$$

To derive the third order three stage TFIRK method, we make the substitution $s = 3, c_1 = 0$ in the recursive relations (5) and (6)

For $i = 2$

$$\cos(c_2 z) - 1 = 0 \quad (10)$$

$$\sin(c_2 z) - za_{2,1} = 0 \quad (11)$$

For $i = 3$

$$\cos(c_3 z) - 1 + za_{3,2}\sin(c_2 z) = 0 \quad (12)$$

$$\sin(c_3 z) - z\big[a_{3,1} + a_{3,2}\cos(c_2 z)\big] = 0 \quad (13)$$

Now, substituting $s = 3$, $c_1 = 0$ in equations (7) and (8)

$$\cos(z) - 1 + zb_{-1}\sin(z) + z[b_2\sin(c_2 z) + b_3\sin(c_3 z)] - z\big[b_2\sin\big((c_2 - 1)z\big) + b_3\sin\big((c_3 - 1)z\big)\big] = 0 \quad (14)$$

$$\sin(z) - zb_1 + zb_{-1}\cos(z) - z[b_2\cos(c_2 z) + b_3\cos c_3 z] + z\big[b_2\cos\big((c_2 - 1)z\big) + b_3\cos\big((c_3 - 1)z\big)\big] = 0 \quad (15)$$

Equations $(12) - (15)$ are now the equations of order conditions for third order three stage trigonometrically fitted method that replaces the order conditions of the original method presented in (9).
To obtain the coefficients of the method we solve the system of two equations, (14) and (15) together with an additional equation from the order condition (9) namely,

$$b_1 - b_{-1} = 1 \quad (16)$$

$$b_{-1} + b_2 + b_3 = \frac{1}{2} \quad (17)$$

These sum up to four equations with six unknown $(b_{-1}, b_1, b_2, b_3, c_2$ and $c_3)$. The equations are solved in terms of two free parameters $\left(c_2 = \frac{31}{60}, c_3 = \frac{62}{85}\right)$ whose values are obtained from the Butcher Tableau of IRK3-3 methods presented in Table I. Equations (16) and (17) are chosen to augment the updated (14) and (15) so that $(b_{-1}, b_1, b_2$ and $b_3)$ are not taken as free parameters.

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{2}\frac{M_1}{M_2} \\ b_1 &= -\frac{1}{2}\frac{M_3}{M_4} \\ b_2 &= \frac{1}{2}\frac{M_5}{M_6} \\ b_3 &= -\frac{1}{2}\frac{M_7}{M_8} \end{aligned}\right\} \qquad (18)$$

where

$$M_1 = z\cos(z)\sin\left(\tfrac{1}{2}z\right) - z\sin(z) + z\sin\left(\tfrac{1}{2}z\right) + \cos(z^2) - 2\cos(z) + 1 + \sin(z^2) - 2\sin(z)\sin\left(\tfrac{1}{2}z\right) \qquad (19)$$

$$M_2 = z(\cos(z) - 1)\left(\sin(z) - 2\sin\left(\frac{1}{2}z\right)\right) \qquad (20)$$

$$M_3 = -2z\sin(z)\cos(z) + 5z\cos(z)\sin\left(\frac{1}{2}z\right) + z\sin(z) - 3z\sin\left(\tfrac{1}{2}z\right) + \cos(z^2) - 2\cos(z) + 1 + \sin(z^2) - 2\sin(z)\sin\left(\tfrac{1}{2}z\right) \qquad (21)$$

$$M_4 = z(\cos(z) - 1)\left(\sin(z) - 2\sin\left(\frac{1}{2}z\right)\right) \qquad (22)$$

$$M_5 = z\sin(z) + 2\cos(z) - 2 \qquad (23)$$

$$M_6 = z\left(\sin(z) - 2\sin\left(\frac{1}{2}z\right)\right) \qquad (24)$$

$$M_7 = \cos(z^2) - 2\cos(z) + z\sin(z) + 1 + z\cos(z)\sin\left(\tfrac{1}{2}z\right) - 3z\sin\left(\tfrac{1}{2}z\right) - \sin(z^2) + 2\sin(z)\sin\left(\tfrac{1}{2}z\right) \qquad (25)$$

$$M_8 = z(\cos(z) - 1)\left(\sin(z) - 2\sin\left(\frac{1}{2}z\right)\right) \qquad (26)$$

Convergence Analysis

It is important to note that the original method, that is, IRK3-3 method needs to be recovered as z approaches zero. As such, the Taylor series expansions of the coefficients $b_{-1}, b_1, b_2,$ and b_3 in (18) are obtained as

$$\left.\begin{aligned} b_{-1} &= -\frac{1}{12} - \frac{7}{1440}z^2 + o(z^4) \\ b_1 &= \frac{11}{12} - \frac{7}{1440}z^2 + o(z^4) \\ b_2 &= \frac{1}{3} - \frac{1}{720}z^2 - \frac{1}{80640}z^4 + o(z^6) \\ b_3 &= \frac{1}{4} + \frac{1}{160}z^2 + o(z^4) \end{aligned}\right\} \qquad (27)$$

From (27) it is clear that as z approaches zero the original method IRK3-3 is recovered.

Next is to verify the order of the method. To check if the method is order three as claimed, we substitute the coefficients of the method into order conditions (9) up to order three and take the Taylor series expansion of each to obtain

$$\left.\begin{aligned} &\text{order 1 :} && b_1 - b_{-1} = 1 + o(z^4) \\ &\text{order 2 :} && b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2} + o(z^4) \\ &\text{order 3 :} && \sum_{i=2}^{s} b_i\, c_i = \frac{5}{12} + \frac{1}{180}z^2 + o(z^4) \end{aligned}\right\} \qquad (28)$$

From (28) as z tends to zero the order conditions of the Improved Runge-Kutta method up to order three are recovered, which implies that the coefficients of Trigonometrically-fitted third order three stage method satisfies the IRK order three conditions.

RESULTS AND DISCUSSION

Numerical results of TFIRK3-3 applied to problems 1-5 with different step sizes and integration intervals are presented in Tables II – VI. The intervals of integration are taken to be 100 and 200. Both small and large integration intervals are considered to measure the stability of the method when solving highly oscillatory problems.

Problem 1 (Homogeneous problem)
$y'(x) = -2\cos(8x) - 8\sin(8x),\ \ y(0) = 1,$ and the fitted frequency $\omega = 8$

Exact solution: $y(x) = -\frac{1}{4}\sin(8x) + \cos(8x)$

Source: Senu *et al*. (2009)

Problem 2
$y'(x) = \cos(x),\ y(0) = 0,\ \omega = 1$

Exact solution: $y(x) = \sin(x)$

Source: Dormand *et al*. (1996)

Problem 3 (Inhomogeneous problem)

$y'(x) = \cos(x) - \sin(x) + 1, y(0) = 1,\ \omega = 1$

Exact solution: $y(x) = \sin(x) + \cos(x) + x$

Source: Al-khasawneh *et al*. 2007

Problem 4

$y'(x) = 2\cos(2x), y(0) = 0, \quad \omega = 2$

Exact solution: $y(x) = \sin(2x)$

Source: Kasim *et al*. 2015

Problem 5 (Homogeneous problem)
$y'(x) = -2\cos(10x) - 10\sin(10x), \quad y(0) = 1, \omega = 10$

Exact solution: $y(x) = -\frac{1}{5}\sin(10x) + \cos(10x)$

Source: Senu *et al.* 2009.

Table II maximum errors of **TFIRK3-3** and **IRK3-3** for **problem 1** with $\omega = 8$

h	Method	100	NFEs	200	NFEs
$\frac{1}{20}$	TFIRK3-3	$9.3660000000\times 10^{-27}$	6000	$9.3740000000\times 10^{-27}$	12000
	IRK3-3	$2.7788280559\times 10^{-04}$	6000	$2.7788281310\times 10^{-04}$	12000
$\frac{1}{40}$	TFIRK3-3	$3.0084400000\times 10^{-25}$	12000	$3.0084400000\times 10^{-25}$	24000
	IRK3-3	$1.7761677402\times 10^{-05}$	12000	$1.7761677402\times 10^{-05}$	24000
$\frac{1}{80}$	TFIRK3-3	$8.6135090000\times 10^{-24}$	24000	$8.6135090000\times 10^{-24}$	48000
	IRK3-3	$1.1199891883\times 10^{-06}$	16000	$1.1199891904\times 10^{-06}$	48000

Table III Maximum errors of TFIRK3-3 and IRK3-3 for problem 2 with $\omega = 1$

h	Method	100	NFEs	200	NFEs
$\frac{1}{20}$	TFIRK3-3	$4.2263157370\times 10^{-22}$	6000	$4.2263157370\times 10^{-22}$	12000
	IRK3-3	$3.5643932006\times 10^{-08}$	6000	$3.5643935071\times 10^{-08}$	12000
$\frac{1}{40}$	TFIRK3-3	$1.0734211713\times 10^{-20}$	12000	$1.0734219515\times 10^{-20}$	24000
	IRK3-3	$2.1989518187\times 10^{-09}$	12000	$2.1989518188\times 10^{-09}$	24000
$\frac{1}{80}$	TFIRK3-3	$5.9559291864\times 10^{-19}$	24000	$5.9559291864\times 10^{-19}$	48000
	IRK3-3	$1.3653424903\times 10^{-10}$	24000	$1.3653424903\times 10^{-10}$	48000

Table IV Maximum errors of TFIRK3-3 and IRK3-3 for problem 3 with $\omega = 1$

h	Method	100	NFEs	200	NFEs
$\frac{1}{20}$	TFIRK3-3	$1.6174317000\times 10^{-21}$	6000	$2.7875850000\times 10^{-21}$	12000
	IRK3-3	$8.4735653434\times 10^{-08}$	6000	$8.4735653434\times 10^{-08}$	12000
$\frac{1}{40}$	TFIRK3-3	$6.9814438400\times 10^{-19}$	12000	$1.3947720310\times 10^{-18}$	24000
	IRK3-3	$5.2677758618\times 10^{-09}$	12000	$5.2677888558\times 10^{-09}$	24000
$\frac{1}{80}$	TFIRK3-3	$3.1929235780\times 10^{-17}$	24000	$6.3918365678\times 10^{-17}$	48000
	IRK3-3	$3.2834602772\times 10^{-10}$	24000	$3.2834608592\times 10^{-10}$	48000

Table V Maximum errors of TFIRK3-3 and IRK3-3 for problem 4 with $\omega = 2$

h	Method	100	NFEs	200	NFEs
$\frac{1}{20}$	TFIRK3-3	$9.0308640000\times 10^{-24}$	6000	$9.0308640000\times 10^{-24}$	12000
	IRK3-3	$5.8501579254\times 10^{-07}$	6000	$5.8502330957\times 10^{-07}$	12000
$\frac{1}{40}$	TFIRK3-3	$4.2263157870\times 10^{-22}$	12000	$4.2263157870\times 10^{-22}$	24000
	IRK3-3	$3.5643935071\times 10^{-08}$	12000	$3.5643935071\times 10^{-08}$	24000
$\frac{1}{80}$	TFIRK3-3	$1.0734219523\times 10^{-20}$	24000	$1.0734219523\times 10^{-20}$	48000
	IRK3-3	$2.1989518188\times 10^{-09}$	24000	$2.1989526444\times 10^{-09}$	48000

Table VI Maximum errors of TFIRK3-3 and IRK3-3 for problem 5 with $\omega = 10$

h	Method	100	NFEs	200	NFEs
$\frac{1}{20}$	TFIRK3-3	$9.8300000000\times 10^{-28}$	6000	$1.1080000000\times 10^{-27}$	12000
	IRK3-3	$6.7000921829\times 10^{-04}$	6000	$6.7000928259\times 10^{-04}$	12000
$\frac{1}{40}$	TFIRK3-3	$4.7781000000\times 10^{-26}$	12000	$4.7803000000\times 10^{-26}$	24000
	IRK3-3	$4.3052030611\times 10^{-05}$	12000	$4.3052030611\times 10^{-05}$	24000
$\frac{1}{80}$	TFIRK3-3	$2.3634560000\times 10^{-24}$	24000	$2.3634560000\times 10^{-24}$	48000
	IRK3-3	$2.7183122806\times 10^{-06}$	24000	$2.7183123207\times 10^{-06}$	48000

Tables II to VI depict the results of solving Problems 1 to 5 using various forms of step length h. When $h = \frac{1}{20}$ with 100 grid points as well as 200 grid points, the TFIRK3-3 method exhibits less errors than the non-fitted IRK3-3 method with the same number of function evaluations (NFEs). The same scenario occurs when h is reduced to $\frac{1}{40}$ and $\frac{1}{80}$ respectively.

CONCLUSION

A three stage third order trigonometrically fitted Improved Runge-Kutta (TFIRK3-3) method for numerical integration of IVPs with oscillatory solutions has been derived. The order of the non-fitted IRK3-3 method was recovered from the Taylor series analysis of the new method; hence its convergence. Comparison of numerical results obtained showed that the TFIRK3-3 method is more effective and efficient than the IRK3-3 method.

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