



Application of Finite Difference Method for the Numerical Solution of Elliptic Equations

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ABSTRACT

In this paper, finite difference method (FDM) is applied to obtain the numerical solution of elliptic equations. The standard discrete finite difference schemes (forward, central and backward) are used for difference approximations and resulting equations are compared with different mesh sizes. The tri-diagonal matrices derived were solved using MATLAB. The results obtained show that Finite Difference Method is very efficient in solving elliptic equations.

Keywords: Elliptic equation, Laplace's equation, discretized equation, temperature

INTRODUCTION

Elliptic partial differential equations represent phenomena that have already reached a steady state and hence are time independent. In elliptic problems, one is given a partial differential equation (PDE) to be satisfied in the interior of a region R , on whose boundary δR additional boundary conditions are also to be satisfied by the solution. In the one-dimensional analogue of ordinary differential equations (ODE's), the region is an interval whose boundary consists of two end-points. Therefore' two-endpoint problems of ODE's may be regarded as boundary value problems of elliptic type. The Numerical Solution of Elliptic Equation Garrett B. (2018).

Randall J. L.(2007) used finite difference approximation for approximating the solution of second and higher order ordinary and partial differential equation. In their work, Taylors series expansion was employed to discretize the order of derivatives required to obtain approximate solution to differential equation (either ordinary or partial) base on the given conditions.

Also Sangeev Kumar summarized different finite difference schemes for the numerical solution of partial differential equations (PDE) using finite difference method (FDM) with both uniform and non-uniform grids were presented. He mainly focuses on the solution of one and two dimensional of partial differential equation. The solution of one dimensional (P.D.E) used explicit, implicit and Crank-Nicolson scheme. While two

dimensional partial differential equation (P.D.E) using alternate direction implicit (A.D.I) alternate directions explicit (A.D.E) was discussed. These discusses extensively on the study of finite difference method.

Dolicanin, C.B., Nicolic, V.B. and Dolicanin C. D. (2010) they said that finite difference technique is used to study of phenomenon in the theory of thin plates. FDM based on replacing differential equation into difference equation. This method can efficiently solve the problem of bending of thin plates. It is used to find solutions for the plate deflection, moments, stress, strain etc.

According to Islam, M..R. and Alias, N. (2010) Finite difference method is used to discretise a parabolic partial differential equation. They introduced a scientific reenactment demonstrate utilizing one dimensional illustrative condition. This model is in regards to dampness and temperature conduct of tropical herbs amid parchedness. Here Jacobi, Gauss-Seidal and Red dark Gauss-Seidal iterative strategies are examined. It has demonstrated that lack of hydration show is fit to recreate mass and temperature dispersion through numerical techniques approach. This scientific reproduction is tedious and skilled to lessen the danger of

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genuine analyses in real process.

George (2015) studied two methods finite difference method (FDM) and finite element method (FEM) for solving two dimensional linear elliptic partial differential equations. These two methods have same accuracy with regular boundaries. But if boundaries are irregular then he concluded that FEM gives better approximation.

Chaudhari and Patel (2015) in their paper, proposed finite element method for numerical solution of nonlinear Poisson’s equation with initial and bound conditions. A specific problem of Poisson’s equation under certain assumption is discussed. It has been shown that increase in the number of grid points gives rise to an increase in accuracy of solution. Finite difference method will have difficulties due to domain with complex geometries; therefore finite element method becomes more evident.

METHODOLOGY

In this paper, we consider elliptic partial differential equation of two dimensions. Suppose that R is a limited district in the (x, y) plane with limit δR ,

the condition:

$$a(x, y) \frac{\delta^2 u}{\delta x^2} + b(x, y) \frac{\delta^2 u}{\delta x \delta y} + c(x, y) \frac{\delta^2 u}{\delta y^2} = d(x, y, u, \frac{\delta u}{\delta x}, \frac{\delta u}{\delta y}) \tag{2.1}$$

where a, b, c, d, and e are functions of the independent variables x and y , is said to be elliptic if $b^2 - 4ac < 0$ for all focuses (x, y) in R .

Finite difference method offers a more straightforward way to deal with the numerical arrangement of fractional differential conditions. A limited distinction technique requires greater consistency of the arrangement and the triangulation (for instance uniform lattice).

In the Finite difference technique, subordinate in halfway differential equations are supplanted by their Finite difference reciprocals. These subordinates are approximated by different blend of the obscure variable happening at framework focuses encompassing the area at which a subsidiary is required.

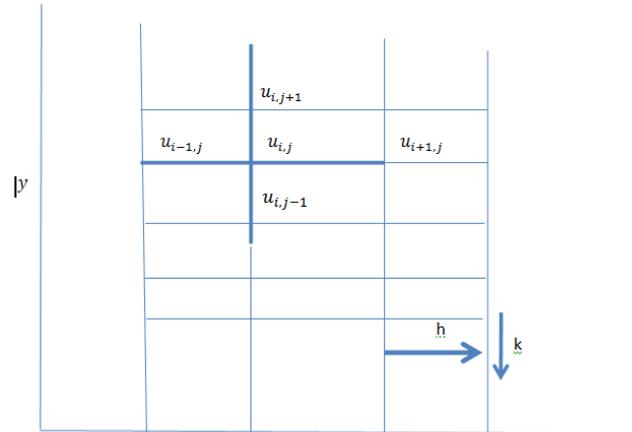


Figure 2.1 Two dimensional Cartesian showing relationship of i and j

The figure 2.1 is a two dimensional Cartesian divided into rectangle grids of width “ h ” and height “ k ” the dependent variable u is a function of two independent variable x and y . The point of derivatives is given in subscripts i, j , where i increases in the x -direction and j increases in the y -direction.

The basis functions for the derivation of the discrete finite difference method are forward, backward, central displacement and averaging operators given by

$$\begin{aligned} \Delta u(x) &= u(x + h) - u(x), \\ \nabla u(x) &= u(x) - u(x - h), \\ \delta u(x) &= u\left(x + \frac{h}{2}\right) - u\left(x - \frac{h}{2}\right), \end{aligned}$$

and

$$\mu u(x) = \frac{1}{2} \left[u\left(x + \frac{1}{2}\right) + u\left(x - \frac{1}{2}\right) \right]$$

respectively, where “ h ” is the length of the interval and $i = 1, 2, 3, 4, 5 \dots \dots \dots n$

Using Taylor’s series expansion;

$$u = u_0 + (x - x_0)u'(x_0) + \frac{(x - x_0)^2}{2!}u''(x_0) + \frac{(x - x_0)^3}{3!}u'''(x_0) \dots \dots \dots$$

let $x - x_0 = h$

$$u(x_i + h) = u(x_i) + hu'(x_i) + \frac{h^2 u''(x_i)}{2!} + \frac{h^3 u'''(x_i)}{3!} + \dots \dots \dots \tag{2.2}$$

$$u(x_i - h) = u(x_i) - hu'(x_i) + \frac{h^2 u''(x_i)}{2!} - \frac{h^3 u'''(x_i)}{3!} + \dots \dots \dots \tag{2-3}$$

Adding equation (2.2) and (2.3), neglecting terms in h^3 and simplify we have

$$u'(x_i) = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} \tag{2.4}$$

Similarly, subtracting equation (2.3) from (2-2) and simplify, we have

$$u'(x_i) = \frac{u_{i+1} + u_{i-1}}{2h} \tag{2.5}$$

For the purpose of this research, we will be using “finite point molecule” to the elliptic equations and we shall focus only on Laplace equation.

To obtain solution of Laplace’s equation using finite difference method we used the following steps;

Consider the Laplace equation:

$$\nabla^2 u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \tag{2.5}$$

Step I Discretize equation (3.24) we have

$$\frac{\delta^2 u}{\delta x^2} = \frac{u_{i-1,j} - 2u_{i,j} + 2u_{i+1,j}}{\nabla x^2} \tag{2.6}$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{u_{i,j+1} - 2u_{i,j} + 2u_{i,j-1}}{\nabla y^2} \tag{2.7}$$

Substituting equation 2.6 and 2.7 into the equation 2.5 to obtain

$$\frac{u_{i-1,j} - 2u_{i,j} + 2u_{i+1,j}}{\nabla x^2} + \frac{u_{i,j+1} - 2u_{i,j} + 2u_{i,j-1}}{\nabla y^2} = 0 \tag{2.8}$$

let $\Delta x = \Delta y = h$

$$\therefore \frac{1}{h^2} (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}) = 0 \tag{2-9}$$

Step II. Form the matrix from the unknown nodes in the grid using the discretized equation.

Step III. Solve the matrix to get the unknowns. Here we used MATLAB to solve the matrix.

NUMERICAL EXAMPLES

1. Consider the torsion problem with different mesh size given by

$$\nabla^2 U(x, y) = -2 \tag{3.1}$$

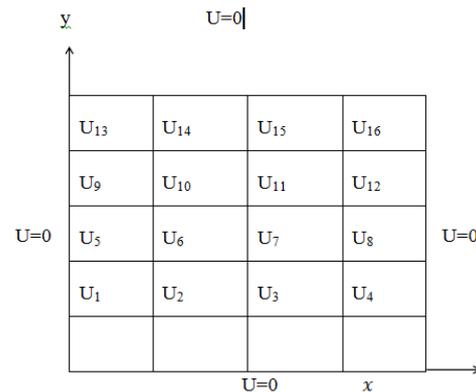
On a unit square domain in $0 < x \leq 1, 0 < y \leq 1$ first quadrant

Taking $h = k = 1/5$

The given problem is equivalent to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2$$

Representing the equation using the nodes we have,



Applying (2.9) we have

$$\frac{u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1}}{h^2} = -2 \tag{3.2}$$

$$U_{i-1,j} + U_{i+1,j} - 4U_{i,j} + U_{i,j-1} + U_{i,j+1} = -2h^2$$

with $h = \frac{1}{5}$ we have

$$U_{i-1,j} + U_{i+1,j} - 4U_{i,j} + U_{i,j-1} + U_{i,j+1} = -2\left(\frac{1}{5}\right)^2$$

Hence

$$u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1} = -\frac{2}{25} \tag{3.3}$$

Using the equation (3.3) to obtain sixteen simultaneous equations presented in the matrix form $MU = b$ below with all the boundary is equal to zero.

$$u = 0 \left(\begin{array}{c|cccc} T_{03} & & U_{13} & U_{23} & U_{33} \\ \hline T_{02} & & U_{12} & U_{22} & U_{32} \\ \hline T_{01} & & U_{11} & U_{21} & U_{31} \\ \hline T_{00} & & & & \end{array} \right) u=0$$

Thus, we have

$$U_{i-1,j} + U_{i+1,j} - 4U_{ij} + U_{ij-1} + U_{ij+1} = 5h^2$$

that is

$$u_{i-1,j} + u_{i+1,j} - 4u_{i,j} + u_{i,j-1} + u_{i,j+1} = \frac{5}{16}$$

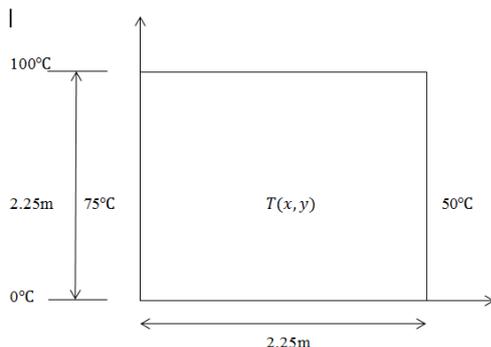
This gives rise to

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ U_{32} \\ U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} = \frac{5}{16} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \\ U_{12} \\ U_{22} \\ U_{32} \\ U_{13} \\ U_{23} \\ U_{33} \end{bmatrix} = \begin{bmatrix} -0.1853 \\ -0.2578 \\ -0.2085 \\ -0.1708 \\ -0.3251 \\ -0.2636 \\ -0.1853 \\ -0.2578 \\ -0.2085 \end{bmatrix}$$

3. We consider application of elliptic equation to temperature distribution. Find the temperature distribution within a square plate of 2.25m x 2.25m with a grid point of 0.75m. The boundary conditions are shown in the figure below.



Here the grid length = 0.75m

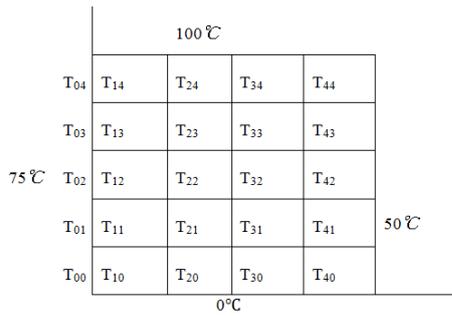
We shall form a $m \times n$ matrix

From the grid length we shall get the number of rows and column. Since the plate is a square plate; then

$$m = n = \frac{2.25}{0.75}$$

$$m = n = 3$$

The plate is represented using the nodes.



The discretized equation is

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{ij} = 0$$

ie $T_{10} = T_{20} = T_{30} = 0$
 $T_{01} = T_{02} = T_{03} = T_{04} = 75^0$
 $T_{14} = T_{24} = T_{34} = 100^0$
 $T_{41} = T_{42} = T_{43} = 50^0$

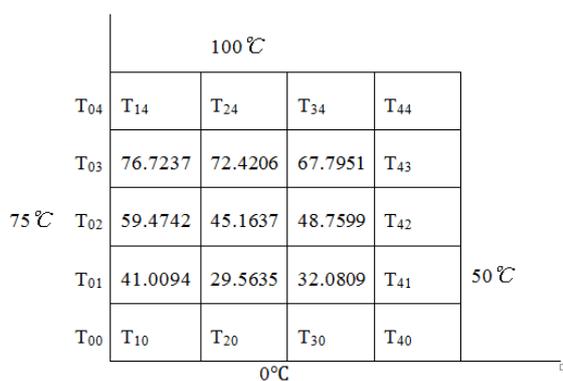
To get the interior nodes the discretized equation is used to obtain the matrix $AT = b$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} = \begin{bmatrix} -75 \\ -75 \\ -175 \\ 0 \\ 0 \\ -100 \\ -50 \\ -50 \\ -150 \end{bmatrix}$$

Hence,

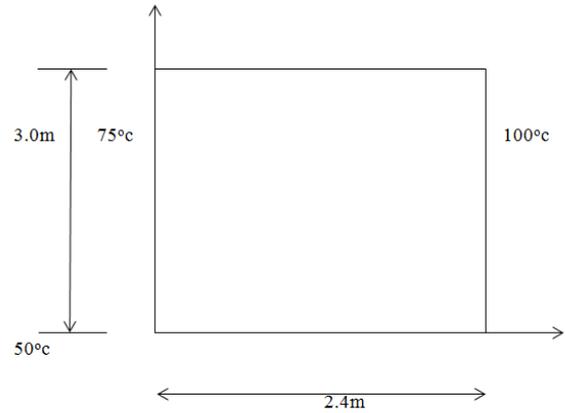
$$\begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} = \begin{bmatrix} 41.0094 \\ 59.4742 \\ 76.7237 \\ 29.5635 \\ 45.1637 \\ 72.4206 \\ 32.0809 \\ 48.7599 \\ 67.7951 \end{bmatrix}$$

Figure 4.1 Showing temperature distribution within a square plate



From figure 4.1 that the unknown nodes increases along the $y - axis$ alone because it is a square plate.

4. A plate $2.4m \times 3.0m$ is subjected to temperature as in the figure below. Use a grid length $0.6m$. Find the temperature of interior nodes.



The grid length = $0.6m$

We shall form a $m \times n$ matrix and it will be gotten by

$$M = \frac{\text{length}}{\text{grid length}}$$

$$M = \frac{3.0m}{0.6m}$$

$$M = 5.0m$$

$$N = \frac{\text{width}}{\text{grid length}}$$

$$N = \frac{2.4m}{0.6m}$$

$$N = 4m$$

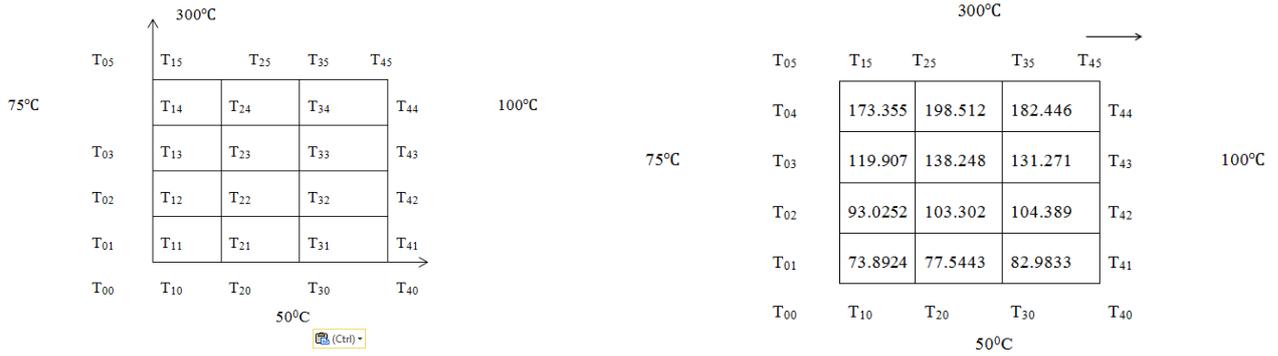
$$dx = dy = 0.6$$

Therefore, the figure above shall be represented with all the interior nodes.

All the nodes on the left to right boundary have an i value = 0

All the nodes on the top to bottom have a j value = 0

Our nodes will be represented in the form T_{ij}



$$T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1} + T_{ij} = 0$$

$$T_{01} = T_{02} = T_{03} = T_{04} = 75^\circ\text{C}$$

$$T_{10} = T_{20} = T_{30} = 50^\circ\text{C}$$

$$T_{41} = T_{42} = T_{43} = T_{44} = 100^\circ\text{C}$$

$$T_{15} = T_{25} = T_{35} = 300^\circ\text{C}$$

The discretized equation is used to get 12 equations with 12 unknown which are the interior nodes as presented in the matrix form $AT = b$ as shown below.

Note that the nodes at the corner i.e T_{05} , T_{45} , T_{40} , T_{00} , are not needed.

Using $AT = b$ to find the unknowns

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{14} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{24} \\ T_{31} \\ T_{32} \\ T_{33} \\ T_{34} \end{bmatrix} = \begin{bmatrix} -125 \\ -75 \\ -75 \\ -375 \\ -50 \\ 0 \\ 0 \\ -300 \\ -150 \\ -100 \\ -100 \\ -400 \end{bmatrix}$$

Solving we have,

$$\begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{14} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{24} \\ T_{31} \\ T_{32} \\ T_{33} \\ T_{34} \end{bmatrix} = \begin{bmatrix} 73.8924 \\ 93.0252 \\ 119.907 \\ 173.355 \\ 77.5443 \\ 103.302 \\ 138.248 \\ 198.512 \\ 82.9833 \\ 104.389 \\ 131.271 \\ 182.446 \end{bmatrix}$$

Figure 4.2 Showing temperature distributions within a rectangular plate

From figure 4.2 we see that temperature distribution increases in both directions of the nodes. It shows that finite difference methods gives regular distribution within the unknown nodes of an elliptic equation. This result is because figure 4.2 is a rectangular plate.

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