



## Zero Dissipative Explicit Hybrid Method for Numerical Integration of Oscillatory Problems

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### ABSTRACT

A zero dissipative and dispersive explicit integrator for oscillatory problems is proposed and derived in this paper. In the derivation, algebraic order conditions as well as dispersion and dissipation conditions are considered. The method has algebraic order five. Numerical experiment is conducted to measure the performance of the new method. Results obtained from the experiment suggest the superiority of the method over several existing methods.

**Keywords:** Zero dissipative, dissipation conditions, Numerical experiment

### INTRODUCTION

In recent time, numerical analysts became interested in finding numerical methods that approximately solve a class of special second-order initial value problem whose solution is oscillatory in nature. This is because only a few of the problems can be solved by analytical means. The problem is given in the equation below:

$$y'' = f(x, y), y(x_0) = y_0, y'(x_0) = y'_0. \quad (1)$$

The differential equation is called special because of its independence on  $y'$  explicitly. The importance of this problem is evident in the attention it has been attracting in the last few decades. Remarkable attempts have been made on the solution procedures of (1). Some of which can found in (Al-Khasawneh, *et al.*, 2007; Bettis, 1973; Butcher, 2008; Dormand, *et al.*, 1980; Franco, 2003; Ming *et al.*, 2012; Mohamed, *et al.*, 2012; Panowsky, *et al.*, 1988; Papadopoulos, *et al.*, 2013; Rabiei, *et al.*, 2012; Ramos, *et al.*, 2007; Senu, *et al.*, 2009; Senu, *et al.*, 2010; Simos, 2002 and Simos, 1999) and the references therein. Research in the solution techniques for (1) continues to grow, because no best technique is found. All attempts are geared towards obtaining methods with relatively better approximations of the solutions. Multistep hybrid methods for solving (1) are proposed, see (Chawla, *et al.*, 1996; Simos, 2012; Tsitouras, 2002 and Vigo-Aguiar, *et al.* 2003). Franco (2006), derived explicit Runge-Kutta related methods using the order

conditions derived in Coleman (2003). Furthermore, Jikantoro *et al.*, (2015), proposed hybrid method with improved error properties. Here, we proposed an explicit hybrid method with more improved numerical properties.

The general form of the method remains:

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^m b_i f(x_n + c_i h, Y_i), i = i \dots m, \\ Y_i = (c_i + 1)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^i a_{ij} f(x_n + c_j h, Y_j). \quad (2)$$

Where  $y_n$  and  $y_{n-1}$  are approximations for  $y(x_n)$  and  $y(x_{n-1})$ , respectively. The parameters  $a_{ij}, b_i, c_i$  are real numbers. Let  $\mathbf{b}$  and  $\mathbf{c}$  be  $m$ -dimensional vectors given by  $\mathbf{b} = [b_1, b_2, \dots, b_m]^T$  and  $\mathbf{c} = [c_1, c_2, \dots, c_m]^T$  and  $\mathbf{A} = [a_{ij}]$  is an  $m \times m$  matrix, then the coefficients of the methods can be summarized in a Butcher-like tableau as follows:

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$$\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^T \end{array}$$

The method (1) is called hybrid method (Franco, 2006), because it is non linear and not self starting. Franco, 2006, derived the coefficients of an explicit class of (2) up to algebraic order six with less computation cost by using the algebraic order conditions of the method developed in (Coleman, 2003). To improve its properties for better performance, Ahmad et al. (2013), proposed semi-implicit hybrid methods up to algebraic order five. In the work of Jikantoro et al. (2015), it is pointed out that there is a huge advantage of improved accuracy and efficiency as the order of this method increases. Hence, we propose a semi-implicit hybrid method with higher algebraic order.

The remaining part of the paper is organized as follows: dispersion and dissipation analysis is presented in Section 2. Derivation of the proposed method is fully presented in Section 3. Stability analysis of the method is presented in Section 4. Results of numerical experiment are presented in Section 5. And Section 6 is conclusion.

### DISPERSION AND DISSIPATION ANALYSIS

To analyze errors due to dispersion and dissipation or phase lag and amplification factor, a simple homogeneous test equation below is considered.

$$y'' = -\lambda^2 y, \lambda > 0 \in R. \tag{3}$$

Apply method (2) on the test problem (3), the following equations are obtained in vector form:

$$\begin{aligned} \mathbf{Y} &= (\mathbf{c} + \mathbf{e})y_n - \mathbf{c}y_{n-1} - v^2 \mathbf{A}\mathbf{Y}, \\ y_{n+1} &= 2y_n - y_{n-1} - v^2 \mathbf{b}^T \mathbf{Y}. \end{aligned} \tag{4}$$

Where  $v = \lambda h$ ,  $\mathbf{e} = [1, 1, \dots, 1]^T$ , vectors  $\mathbf{b}$ ,  $\mathbf{c}$  and matrix  $\mathbf{A}$  are given in Section 1. From (4) we get

$$\mathbf{Y} = (v^2 \mathbf{A} + \mathbf{I})^{-1} (\mathbf{c} + \mathbf{e})y_n - (v^2 \mathbf{A} + \mathbf{I})^{-1} \mathbf{c}y_{n-1}, \tag{5}$$

which when substituted in second part of eqn. (4) we get

$$y_{n+1} - T_1(v^2)y_n + T_2(v^2)y_{n-1} = 0. \tag{6}$$

Where

$$T_1(v^2) = 2 + \sum_{i=1}^{m-1} U_i v^{2i}, T_2(v^2) = 1 + \sum_{i=1}^{m-1} V_i v^{2i}. \tag{7}$$

$V_i$  and  $U_i$  are determined by the coefficients of the method.

It has been by Verde howen and Someijer, 1987, that the true solution of eqn. (6) is  $y_n = 2 |\rho|^n \cos(\omega + n\phi)$ ,  $\tag{8}$

where  $\rho, \omega$  are dependent on the coefficients of the method and  $\rho, \phi$  are amplification factor and phase, respectively. The true solution of eqn. (3) is

$$y(x_n) = 2 |\delta| \cos(\varphi + n\psi), \tag{9}$$

where  $\delta$  and  $\varphi$  are real constants dependent on the initial conditions of the problem, and  $n$  is the number of terms. The definition formulated in (Van der Houwen and Sommeijer, 1983) adopted by (Franco, 2006 and Jikantoro, 2015) follows.

**Definition 1.** From eqns. (8) and (9), the quantity  $R(v) = v - \phi$  is referred to dispersion error or phase lag of the method. The method is said to have dispersion error of order  $q$  if  $R(v) = O(v^{q+1})$ . Furthermore, the quantity  $S(v) = 1 - |\rho|$  is referred to dissipation error of the method. And the method is said to be dissipative of order  $r$  if  $S(v) = O(v^{r+1})$ . [10].

From **Definition 1**, it follows that

$$R(v) = v - \cos^{-1} \left( \frac{T_1(v^2)}{2\sqrt{T_2(v^2)}} \right), S(v) = 1 - \sqrt{T_2(v^2)}. \tag{10}$$

Taking the Taylor expansions of (10) when  $m = 5$  yields the order conditions of dispersion

and dissipation, which are not stated here due to too much space they will occupy.

**STABILITY ANALYSIS**

Like dispersion and dissipation errors, to analyze stability of (2), the method is applied on the test eqn. (3). We have done this in Section 2. Re-writing the eqn. (6) gives stability polynomial of the method as

$$\chi^2 - T_1(v^2)\chi + T_2(v^2) = 0. \tag{11}$$

The numerical solution defined by (6) should be periodic. This is true only if its coefficients satisfy the following conditions:

$$T_2(v^2) \equiv 1, |T_1(v^2)| < 2, \text{ for all } v^2 \in (0, v_p^2),$$

where  $(0, v_p^2)$  is known as interval of periodicity of the method. With this, the method is said to be zero-dissipative since  $S(v) = 0$ , which implies that the order of dissipation of the method is infinity. On the other hand, if the

order of dissipation is finite that is,  $S(v) \neq 0$ , then absolute stability of the method is guaranteed on the following conditions:

$$T_2(v^2) < 1, |T_1(v^2)| < 1 + T_2(v^2), \text{ for all } v^2 \in (0, v_s^2),$$

where  $(0, v_s^2)$  is interval of absolute stability of the method. Detail of this analysis can be found in [6, 10, 26].

**DERIVATION OF THE METHOD**

Presented in this section is the derivation of the proposed method of order five with five stages. The derivation will focus on maximizing the order of dissipation by using the order conditions of dissipation derived in Section 2. The algebraic order conditions of two-step hybrid methods derived by Coleman, 2003, are given below:

**Order 1:**  $\sum b_i = 1,$

**Order 2:**  $\sum b_i c_i = 0,$

**Order 3:**  $\sum b_i c_i^2 = \frac{1}{6}, \sum b_i a_{ij} = \frac{1}{12},$

**Order 4:**  $\sum b_i c_i^3 = 0, \sum b_i c_i a_{ij} = \frac{1}{12}, \sum b_i a_{ij} c_j = 0,$

**Order 5:**  $\sum b_i c_i^4 = \frac{1}{15}, \sum b_i c_i^2 a_{ij} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j = -\frac{1}{60}, \sum b_i a_{ij} a_{ik} = \frac{7}{120}, \sum b_i a_{ij} c_j^2 = \frac{1}{180},$

$$\sum b_i a_{ij} a_{jk} c_k = \frac{1}{360},$$

**Order 6:**  $\sum b_i c_i^5 = 0, \sum b_i c_i^3 a_{ij} = \frac{1}{30}, \sum b_i c_i^2 a_{ij} c_j = 0, \sum b_i c_i a_{ij} a_{ik} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j^2 = \frac{1}{72},$

$$\sum b_i c_i a_{ij} a_{jk} = -\frac{1}{720}, \sum b_i a_{ij} a_{jk} c_k = -\frac{1}{120}, \sum b_i a_{ij} c_j^3 = 0, \sum b_i a_{ij} c_j a_{ik} = \frac{1}{360},$$

$$\sum b_i a_{ij} c_{jk} c_k = 0.$$

The order conditions have the following simplifying assumption:

$$\sum_{j=1}^m a_{ij} c_j^\lambda = \frac{c_i^{\lambda+2} + (-1)^\lambda c_i}{(1+\lambda)(2+\lambda)}, \lambda \geq 0. \tag{12}$$

To derive fifth order zero-dissipative method we consider algebraic order conditions up to order five given above, simplifying assumption (12), where  $\lambda = 0, 1$ , and dissipation conditions up to order nine. The system of fourteen non linear equations obtained is solved with seventeen unknown parameters, meaning that there are three free parameters, which are  $(c_3, c_4, c_5)$ . Choosing the free parameters appropriately, we get the method as presented in Butcher-like tableau below:

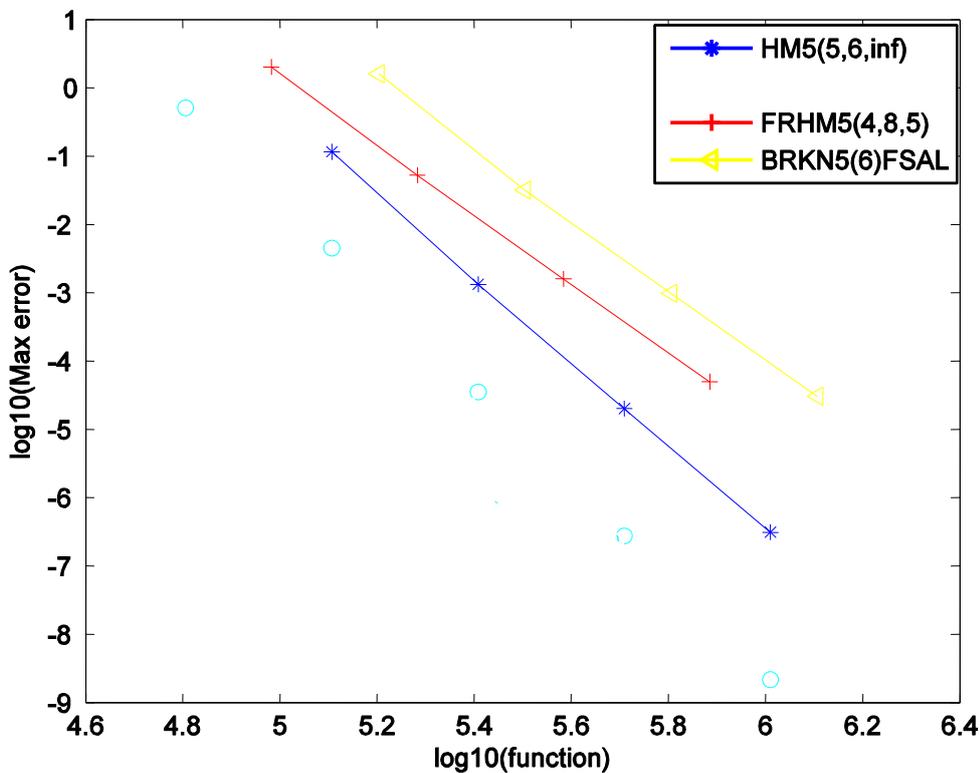
**Table 2:** Coefficients of the proposed method

-1	0				
0	0	0			
$\frac{14}{15}$	$\frac{203}{10125}$	$\frac{8932}{10125}$	0		
$-\frac{2}{3}$	$-\frac{205}{16443}$	$-\frac{601}{3969}$	$\frac{75}{1421}$	0	
$\frac{84}{125}$	$\frac{172787216}{7080078125}$	$\frac{138778052}{244140625}$	$-\frac{8172717}{226562500}$	$\frac{4907301}{976562500}$	0
	$-\frac{11}{1653}$	$\frac{4295}{7056}$	$-\frac{1875}{159152}$	$\frac{813}{4016}$	$\frac{48828125}{235550448}$

**NUMERICAL RESULTS**

**Test Problem**

$$y''(x) = -64y(x), y(0) = 1, y'(0) = -2, y(x) = -\frac{1}{4}\sin(8x) + \cos(8x), x \in (0, 4000).$$



**Figure:** Efficiency curve of HM5 (5, 6, ∞) for the problem

The following acronyms are used in the paper:

- **HM5(5,6, $\infty$ ):** Zero-dissipative fifth order five stage sixth order dispersive hybrid method derived in this paper.
- **HM5(4,8,5):** Fifth order four stage hybrid method with dispersion order eight and dissipation order five in Franco [10].
- **BRKN5(6)FSAL:** Fifth order six stage Runge-Kutta-Nyström method with first same as last property presented in Bettis [3].

Numerical results of fifth order five stage hybrid methods derived are compared with existing methods in the literature. The results are presented in the figure above to demonstrate the efficiency of the proposed method for solving the test problem.

The figure reveals that HM5(5,6, $\infty$ ) is more efficient compared to BRKN5(6)FSAL due to the fact that it has four function evaluation per step while BRKN5(6)FSAL has five

### Conclusion

It is therefore concluded from the results that the methods HM5(5,6, $\infty$ ) is better than the existing methods of the same algebraic order considered here.

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