



On convergence rate of Three-step Iterative Schemes in Banach Spaces

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ABSTRACT

In this article, we compare the three step iterative schemes for a class of Zamfirescu operator in Banach space. We show that for the class of the operator, the CR and SP iterative schemes are faster than Noor and Ishikawa iterative schemes. Moreover, we present various numerical examples using MATLAB program. Our results shown that as far as the rate of convergence is concerned, the CR iterative scheme is the best for increasing functions, while SP iterative scheme is the best for decreasing functions. The result obtained from the numerical computation also confirm the theoretical analysis.

Keywords: Fixed point, Rate of Convergence, Iterative Schemes, CR and SP iterative schemes

INTRODUCTION

Most of the problems that arise in different disciplines of sciences can be formulated by the equation of the form:

$$f(x) = 0 \quad (1.1)$$

where f is some function. Equation (1.1) can easily be reformulated as the fixed point equations of the type:

$$Tx = x \quad (1.2)$$

where T is a self-map of an ambient space X and $x \in X$. These equations are often classified as linear or nonlinear, depending on whether the mappings used in the equation is linear with respect to the variables. Banach fixed-point theorem is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provide a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892-1945), and was first stated by him in 1922. In Mathematics, more specifically in functional analysis, a Banach space (pronounced [banax]) is a complete normed vector space. Thus, a Banach space is a vector space with metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit that is within the space. Over the years, a considerable attention has been paid to solving such equations by using different techniques such as direct and iterative methods. In case of linear equations, both direct and

iterative methods are used to obtain solutions of the equations. But in case of nonlinear equations, due to various reasons, direct method can be impractical or fail in solving equations and thus iterative method become a viable alternative. Nonlinear problems are of importance interest to mathematicians, physicists and engineers and many other scientists, simply because most systems are intrinsically nonlinear in nature. That is why, researchers in various disciplines of sciences are often faced with the solving such problems. It would be hard to fudge that the role of iterative approximation of fixed points have played in the recent progress of nonlinear science. Indeed, the phrase iterative approximation has been introduced to describe repetition- based research into nonlinear problems that are inaccessible to analytic methods. For this reason, the iterative approximation of fixed points has become one of the major and basic tools in the theory of equations and as a result, numerous iterative methods have been introduced or improved and studied for many years in detail from various point of aspect by a wide audience of researchers.

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The origin of fixed point theory lies in the method of the successive approximation used for proving existence solution of differential equations introduced by Charles Emile Picard in 1890. But formally it was started in the beginning of the twentieth century as an important part of analysis. The abstraction of this classical theory is the pioneering work of great Polish mathematicians.

Fixed point iterative procedures are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative procedures. In computational mathematics, it is of interest to know which of the given iterative procedures converge faster to a desired solution. Commonly know as the rate of convergence. Several authors have studied the rate of convergence of various iterative scheme in Banach spaces. There are several process in the literature for which the fixed points of operation have been approximated over the year by various authors. This fact is confirmed by great number of published books and papers.[Mann (1953), Ishikawa (1974), Noor (2000), Agarwal et al (2007), Renu Chug (2002), Wahab et al. (2016) and so forth].

In this paper, we obtain a common iterative scheme for some iterative algorithms for finding the fixed points of Zamfirescu operator and the property of the operator that is sufficient to guarantee convergence of the sequence to a fixed point whenever such fixed point exists. We show that CR iteration method can be used to approximate fixed point of Zamfirescu operator. It is shown that the CR and SP iteration methods converge faster to that of Noor and Ishikawa iterations when applied to Zamfirescu operator.

Preliminaries and Definition of Terms

The following definitions are useful in the prove of our results.

Definition 2.1: Let (X, d) be a metric space. Then, a map $T: X \rightarrow X$ is called a contraction mapping on X if there exists $a \in (0,1)$ such that

$$d(Tx, Ty) \leq ad(x, y) \tag{2.1}$$

$\forall x, y \in X$.

The function d is called distance function.

Definition 2.2: Let $(E, ||. ||)$ be a Banach space. A map $T: X \rightarrow X$ is called a weak-contraction if there exists a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$||Tx - Ty|| \leq \delta ||x - y|| + L ||y - Ty|| \tag{2.2}$$

$\forall x, y \in X$.

Definition 2.3: Let $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ be two sequences of real numbers with limits a and b respectively. Assume that there exists

$$\lim_{n \rightarrow \infty} \frac{|a_n - a|}{|b_n - b|} = l \tag{2.3}$$

(i) If $l = 0$ then we say that $\{a_n\}_{n=0}^\infty$ converges faster than $\{b_n\}_{n=0}^\infty$; and

(ii) If $0 < l < \infty$ then we say that $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ have the same rate of convergence.

Definition 2.4: Assume that for two fixed points iteration processes $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ both converging to the same fixed point q , the following errors are estimated.

$$||u_n - q|| \leq a_n \text{ for all } n \in N$$

$$||v_n - q|| \leq b_n \text{ for all } n \in N$$

are available where $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are two sequences of positive numbers (covering to zero). If $\{a_n\}_{n=0}^\infty$ converges faster than $\{b_n\}_{n=0}^\infty$, then $\{u_n\}_{n=0}^\infty$ converges faster than $\{v_n\}_{n=0}^\infty$ to q .

Definition 2.5: A normed space $(E, ||. ||)$ is called a Banach space if it is complete under the associated metric that is, if each Cauchy sequence converges in E .

Iterative Schemes

In this section, we shall discuss the following iterative schemes: Picard, Mann, Ishikawa, Thiawan, S-iteration, Noor, SP and CR iterative schemes.

Picard Iteration:

In a complete metric space, the Picard iterative scheme $\{x_n\}_{n=0}^\infty$ defined by

$$x_{n+1} = Tx_n, n = 0,1,2 \dots \tag{3.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality

$$d(Tx, Ty) \leq ad(x, y) \quad (3.2)$$

for all $x, y \in X$ and $a \in [0,1)$ condition (3.2) is called the Banach's contraction condition. Any operator satisfying (3.2) is called a strict contraction.

Mann Iteration

In 1953, Mann defined the Mann iterative scheme as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n \quad (3.3)$$

where $\{\alpha_n\}_{n=0}^\infty$ is a sequence of positive numbers in $[0,1]$.

Ishikawa Iteration

In 1974, Ishikawa defined the Ishikawa iterative scheme as:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \quad (3.4) \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$.

S-Iteration

In 2007, Agarwal *et al.* defined the S-iterative scheme as:

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_nTy_n \quad (3.5)$$

$$y_n = (1 - \beta_n)x_n + \beta_nTx_n$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$.

Thianwan Iteration

In 2009, Thianwan defined a new iteration process given by:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \quad (3.6) \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$.

Picard-Mann Iteration

In 2013, Khan introduces another iterative scheme called Picard-Mann iteration process which is defined as:

$$\begin{aligned} x_{n+1} &= Ty_n \quad (3.7) \\ y_n &= (1 - \alpha_n)x_n + \alpha_nTx_n \end{aligned}$$

where $\{\alpha_n\}$ is a sequence of positive numbers in $[0,1]$.

Noor Iteration

In 2000, Noor defined the Noor iterative scheme as:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n \quad (3.8) \\ y_n &= (1 - \beta_n)x_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

SP-Iteration

In 2011, Phuengrattana and Suantai defined the SP iterative scheme as:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \quad (3.9) \\ y_n &= (1 - \beta_n)z_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$.

CR Iteration

In 2012, Renu Chugh defined the CR iterative scheme as:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)y_n + \alpha_nTy_n \quad (3.10) \\ y_n &= (1 - \beta_n)Tx_n + \beta_nTz_n \\ z_n &= (1 - \gamma_n)x_n + \gamma_nTx_n \end{aligned}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum \alpha_n = \infty$.

Theorem 3.1 (Banach): Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a contraction map on X . Then, T has a unique fixed point $x \in X$.

Lemma 3.1: Let δ be a real number such that $\delta \in [0,1)$ and $\{x_n\}_{n=0}^\infty$ is a sequence of non negative numbers such that $\lim_{n \rightarrow \infty} u_n = 0$; then for any sequence of positive numbers $\{u_n\}_{n=0}^\infty$ satisfying

$$x_{n+1} \leq \delta x_n + u_n \quad (3.11)$$

$\forall n \in \mathbb{N}$.

we have

$$\lim_{n \rightarrow \infty} x_n = 0 \tag{3.12}$$

The most generalized operator used to approximate fixed point is the one proved by Zamfirescu (1972). The Zamfirescu operator was obtained from the Banach(1932), Kannan(1968) and Chatterjea (1972) contractive mapping as follows. The operator T is called a Kannan mapping if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \tag{3.13}$$

for all $x, y \in X$.

Another similar definition due to Chartterjea mapping is as follows: if there exists $C \in (0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \tag{3.14}$$

By combining (3.1), (3.7) and (3.8) conditions, we have the Zamfirescu operator

$$\begin{aligned} Z_1 : d(Tx, Ty) &\leq ad(x, y) \\ Z_2 : d(Tx, Ty) &\leq b[d(x, Tx) + d(y, Ty)] \end{aligned} \tag{3.15}$$

$$Z_3: d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

$\forall x, y \in X$.

The Zamfirescu operator was used to prove the strong convergence of (3.1) as follows:

Theorem 3.2: Let C be a nonempty subset of a normed space $(E, || \cdot ||)$. Let $T: C \rightarrow C$ be Zamfirescu operator. If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T .

Proof

By Lemma 3.1, T has a unique fixed point in C , say q . Let $x, y \in C$ since T is a Z - operator, at least one of each conditions (3.9) is satisfied. If Z_2 holds, then

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \tag{3.16}$$

is equivalent to

$$\begin{aligned} ||Tx - Ty|| &\leq b(||x - Tx|| + ||y - Ty||) \\ &\leq b[||x - Tx|| + ||y - x|| + ||x - Tx|| + ||Tx - Ty||] \end{aligned}$$

$$= 2b||x - Tx|| + b||y - x|| + b||Tx - Ty|| \tag{3.17}$$

This implies that

$$(1 - b)||Tx - Ty|| \leq b||x - y|| + 2b||x - Tx|| \tag{3.18}$$

Therefore

$$||Tx - Ty|| \leq \frac{b}{1-b} ||x - y|| + \frac{2b}{1-b} ||x - Tx|| \tag{3.19}$$

Similarly if Z_3 holds, we obtain

$$||Tx - Ty|| \leq \frac{c}{1-c} ||x - y|| + \frac{2c}{1-c} ||x - Tx|| \tag{3.20}$$

Let

$$\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\} \tag{3.21}$$

Then we have $0 \leq \delta < 1$ and in view of (3.21), it results in (3.19) and (3.20) become

$$||Tx - Ty|| \leq \delta ||x - y|| + 2\delta ||x - Tx|| \tag{3.22}$$

for all $x, y \in C$.

Now, for $x = q$ and $y = x_n$ in (3.22), we obtain

$$||Tq - Tx_n|| \leq \delta ||x_n - q|| \tag{3.23}$$

which implies

$$||x_{n+1} - q|| \leq \delta ||x_n - q|| \tag{3.24}$$

By Lemma 3.1, we conclude that x_n converges strongly to q .

RESULTS

We present our main results using Z -operator (3.15) on normed spaces for the three-step iterative schemes defined in (3.8), (3.9) and (3.10).

Throughout, \mathbb{N} denotes the set of all positive integers; C is nonempty convex subset of a complete normed space $(E, || \cdot ||)$ and $T: C \rightarrow C$ is a self-map.

Theorem 4.1 Let C be a nonempty subset of normed space $(E, || \cdot ||)$. Let $T: C \rightarrow C$ be Z -operator (3.15). Let $\{x_n\}$ be defined with iterative process (3.8). If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ Then, $\{x_n\}$ converges strongly to a fixed point of T .

Proof

Let $x_0 \in C$; by using (3.8) and (3.15), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n)x_n + \alpha_nTy_n - (1 - \alpha_n + \alpha_n)q\| \\ &= (1 - \alpha_n)\|(x_n - q) + \alpha_n(Ty_n - q)\| \end{aligned} \tag{4.1}$$

$$\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n\|Ty_n - q\|$$

with $x = q$ and $y = y_n$ in (3.22)

$$\|Ty_n - q\| = \delta\|y_n - q\| \tag{4.2}$$

we have

$$\begin{aligned} \|y_n - q\| &= \|(1 - \beta_n)x_n + \beta_nTz_n - (1 - \beta_n + \beta_n)q\| \\ &\leq (1 - \beta_n)\|x_n - q\| + \beta_n\|Tz_n - q\| \end{aligned} \tag{4.3}$$

if $x = q$ and $y = z_n$ in (3.22),

we find that

$$\|Tz_n - q\| \leq \delta\|z_n - q\| \tag{4.4}$$

Combining (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned} \|Ty_n - q\| &\leq \delta[(1 - \beta_n)\|x_n - q\| + \delta\beta_n\|z_n - q\|] \end{aligned} \tag{4.5}$$

Also

$$\begin{aligned} \|z_n - q\| &= \|(1 - \gamma_n)x_n + \gamma_nTx_n - (1 - \gamma_n + \gamma_n)q\| \\ &\leq (1 - \gamma_n)\|x_n - q\| + \gamma_n\|Tx_n - q\| \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$\begin{aligned} \|Ty_n - q\| &\leq \delta[(1 - \beta_n)\|x_n - q\| + \delta\beta_n[(1 - \gamma_n)\|x_n - q\| + \gamma_n\|Tx_n - q\|]] \end{aligned} \tag{4.7}$$

again from (27) if $x = q$ and $y = x_n$ we have

$$\|Tx_n - q\| \leq \delta\|x_n - q\| \tag{4.8}$$

Now combining (4.6), (4.7) and (4.8) we obtain

$$\begin{aligned} \|x_{n+1} - q\| &\leq \|x_n - q\| + \alpha_n[\delta(1 - \beta_n)\|x_n - q\| + \delta\beta_n[(1 - \gamma_n)\|x_n - q\| + \gamma_n\delta\|Tx_n - q\|]] \\ &= [1 - (1 - \beta_n)(1 + \delta\beta_n + \delta^2\beta_n\gamma_n)\alpha_n]\|x_n - q\| \end{aligned} \tag{4.9}$$

since, $[1 - (1 - \beta_n)(1 + \delta\beta_n + \delta^2\beta_n\gamma_n)\alpha_n] \leq [1(1 - \delta)\alpha_n]$ so, from (4.9)

$$\|x_{n+1} - q\| \leq [1(1 - \delta)\alpha_n]\|x_n - q\| \tag{4.10}$$

and using the fact that $0 \leq \delta < 1$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Now, from (4.9) we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - q\| = 0$$

therefore $\{x_n\}_{n=0}^{\infty}$, converges strongly to the fixed point of q .

This complete the proof.

Theorem 4.2: Let C be a nonempty subset of normed space $(E, \|\cdot\|)$. Let $T: C \rightarrow C$ be Z-operator (3.5). Let $\{x_n\}$ be defined with iterative process (3.9). If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a fixed point point of T .

Proof

Let $x_0 \in C$ by using (3.9) and (3.15), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n)y_n + \alpha_nTy_n - q\| \\ &= \|(1 - \alpha_n)(y_n - q) + \alpha_n(Ty_n - q)\| \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\leq (1 - \alpha_n)\|y_n - q\| + \alpha_n\delta\|y_n - q\| \\ &\leq [(1 - \alpha_n)] + \alpha_n\delta\|y_n - q\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - q\| &= \|(1 - \beta_n)z_n + \beta_nTz_n - q\| \\ &= \|(1 - \beta_n)z_n - (1 - \beta_n)q + \beta_n(Tz_n - q)\| \\ &= \|(1 - \beta_n)(z_n - q) + \beta_n(Tz_n - q)\| \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\leq (1 - \beta_n)\|z_n - q\| + \beta_n\delta\|z_n - q\| \\ &\leq [(1 - \beta_n)] + \beta_n\delta\|z_n - q\| \end{aligned}$$

also

$$\begin{aligned} \|z_n - q\| &= \|(1 - \gamma_n)x_n + \gamma_nTx_n - q\| \\ &= \|(1 - \gamma_n)(x_n - q) + \gamma_n(Tx_n - q)\| \\ &\leq (1 - \gamma_n)\|x_n - q\| + \gamma_n\delta\|x_n - q\| \\ &\leq [(1 - \gamma_n)] + \gamma_n\delta\|x_n - q\| \end{aligned} \tag{4.13}$$

substituting (4.12) and (4.13) in (4.11), we have

$$\|x_{n+1} - q\| \leq (1 - \alpha_n) + \alpha_n\delta\{(1 - \beta_n) + \beta_n\delta\|z_n - q\|\}$$

$$\begin{aligned} &\leq (1 - \alpha_n) + \alpha_n \delta (1 - \beta_n) + \beta_n \delta \{ (1 - \gamma_n) \\ &\quad + \gamma_n \delta |x_n - q| \} \\ &= (1 - \alpha_n) + \alpha_n \delta \{ (1 - \beta_n) + \\ &\beta_n \delta \{ (1 - \gamma_n) + \gamma_n \delta |x_n - q| \} \} \\ &= (1 - \alpha_n) + \alpha_n \delta |x_n - q| \\ &= [1 - (1 - \delta) \alpha_n] |x_n - q| \quad (4.14) \end{aligned}$$

$n = 0, 1, 2, \dots$

and using the fact that $0 \leq \delta < 1$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

we obtain that

$$\lim_{n \rightarrow \infty} |x_{n+1} - q| = 0$$

Hence, $\{\alpha_n\}_{n=0}^{\infty}$, converges strongly to the fixed point of q .

This complete the proof.

Theorem 4.3: Let C be a nonempty subset of normed space $(E, \|\cdot\|)$. Let $T: C \rightarrow C$ be Z-operator (3.5). Let $\{x_n\}$ be defined with iterative process (3.10). If $F(T) \neq \emptyset, \sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a fixed point of T

Proof

Let $x_0 \in C$; by using (3.10) and (3.15), we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(1 - \alpha_n)y_n + \alpha_n T y_n - q\| \\ &= \|(1 - \alpha_n)y_n - (1 - \alpha_n)q + \alpha_n(T y_n - q)\| \\ &= \|(1 - \alpha_n)(y_n - q) + \alpha_n(T y_n - q)\| \\ &\leq (1 - \alpha_n) \|y_n - q\| + \alpha_n \|T y_n - q\| \quad (4.15) \\ &\leq (1 - \alpha_n) \|y_n - q\| + \alpha_n \delta \|y_n - q\| \\ &\leq (1 - \alpha_n) + \alpha_n \delta \|y_n - q\| \end{aligned}$$

and

$$\begin{aligned} \|y_n - q\| &= \|(1 - \beta_n)T x_n + \beta_n T z_n - q\| \\ &= \|(1 - \beta_n)T x_n - (1 - \beta_n)q + \beta_n(T z_n - q)\| \\ &= \|(1 - \beta_n)(T x_n - q) + \beta_n(T z_n - q)\| \\ &\leq (1 - \beta_n) \|T x_n - q\| + \beta_n \|T z_n - q\| \quad (4.16) \\ &\leq (1 - \beta_n) \|T x_n - q\| + \beta_n \delta \|z_n - q\| \end{aligned}$$

$$\leq (1 - \beta_n) \delta \|x_n - q\| + \beta_n \delta \|z_n - q\|$$

also

$$\begin{aligned} \|z_n - q\| &= \|(1 - \gamma_n)x_n + \gamma_n T x_n - q\| \\ &= \|(1 - \gamma_n)x_n - (1 - \gamma_n)q + \gamma_n(T x_n - q)\| \\ &= \|(1 - \gamma_n)(x_n - q) + \gamma_n(T x_n - q)\| \quad (4.17) \end{aligned}$$

$$\begin{aligned} &\leq (1 - \gamma_n) \|x_n - q\| + \gamma_n \|T x_n - q\| \\ &\leq (1 - \gamma_n) \|x_n - q\| + \gamma_n \delta \|x_n - q\| \\ &= [(1 - \gamma_n)] + \gamma_n \delta \|z_n - q\| \end{aligned}$$

substituting (4.16), (4.17) in (4.15), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1 - \alpha_n) \\ &\quad + \alpha_n \delta (1 - \beta_n) \delta \|x_n - q\| \\ &\quad + \beta_n \delta \|z_n - q\| \\ &= (1 - \alpha_n) + \alpha_n \delta^2 - \alpha_n \beta_n \delta^2 \|x_n - q\| \\ &\quad + \beta_n \delta (1 - \gamma_n) \\ &\quad + \gamma_n \delta \|x_n - q\| \end{aligned}$$

$$= (1 - \alpha_n) + \alpha_n \delta^2 - \alpha_n \beta_n \delta^2 + \alpha_n \beta_n \delta^2 - \alpha_n \beta_n \gamma_n \delta^2 + \alpha_n \beta_n \gamma_n \delta^3 \|x_n - q\|$$

$$\leq \delta \{ 1 - \alpha_n (1 - \delta) - (1 - \delta) (1 - \alpha_n) \beta_n \gamma_n - \delta (1 - \delta) \alpha_n \beta_n \gamma_n \} \|x_n - q\| \quad (4.18)$$

Since $0 \leq \delta < 1, \sum_{n=0}^{\infty} \alpha_n = \infty$ we get

$$\lim_{n \rightarrow \infty} |x_{n+1} - q| = 0$$

therefore,

$$x_{n+1} \rightarrow q \in F(T)$$

Numerical Results

In this section, we compare the rate of convergence of Noor, SP, CR and Ishikawa iteration procedure with the aid of MATLAB for two examples. The outcome is listed in the Tables . By taking initial approximation of $x_0 = 0.8$ and $\alpha_n = \beta_n = \gamma_n = \frac{1}{(1+n)^2}$, for all iterative schemes.

Example 1: (decreasing function)

Let $f: [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = (1 - x)^7$ with fixed point $q = 0.203455645871543$. The comparison of convergence iterative scheme results is as shown in the Table below:

n	CR iteration	Noor iteration	SP iteration	Ishikawa iteration
0	0.8000000000000000	0.8000000000000000	0.8000000000000000	0.8000000000000000
1	0.346942524167518	0.271515217945557	0.231776580559820	0.068640852035122
2	0.228098457123870	0.183833881391211	0.197978410300222	0.511557006338186
3	0.209132253876284	0.210519206215124	0.203802287356444	0.132863854888163
4	0.204612308621262	0.203292177087995	0.203450452033272	0.296598271855019
5	0.203611614346713	0.203441629722156	0.203450452033272	0.159223365080106
6	0.203464733440823	0.203453163688457	0.203455645869415	0.243693685029081
7	0.203455494688090	0.203455041423507	0.203455645871543	0.184230260972177
8	0.203455659058752	0.203455467893796	0.203455645871543	0.215447565395366
9	0.203455643858705	0.203455586247972	0.203455645871543	0.198148858704851
10	0.203455646299804	0.203455623914314	0.203455645871543	0.205823839537461
11	0.203455645756810	0.203455637172544	0.202580692553319
12	0.203455645908084	0.203455642217475	0.203748618486190
13	0.203455645858206	0.203455644260832	0.203370632751147
14	0.203455645876982	0.203455645132138	0.203477246846968
15	0.203455645869110	0.203455645520105	0.203450900808882
16	0.203455645872721	0.203455645699377	0.203456539723941
17	0.203455645870933	0.203455645784927	0.203455503420236
18	0.203455645871878	0.203455645826924	0.203455664741992
19	0.203455645871349	0.203455645848065	0.203455643844562
20	0.203455645871661	0.203455645858950	0.203455646041821
21	0.203455645871468	0.203455645864669	0.203455645860975
22	0.203455645871592	0.203455645867731	0.203455645871981
23	0.203455645871509	0.203455645869397	0.203455645871533
24	0.203455645871567	0.203455645870319	0.203455645871543
25	0.203455645871526	0.203455645870836	0.203455645871543
26	0.203455645871556	0.203455645871130	0.203455645871543
27	0.203455645871533	0.203455645871299
28	0.203455645871550	0.203455645871397

29	0.203455645871537	0.203455645871455
30	0.203455645871548	0.203455645871489
31	0.203455645871539	0.203455645871510
32	0.203455645871546	0.203455645871523
33	0.203455645871540	0.203455645871530
34	0.203455645871545	0.203455645871535
35	0.203455645871541	0.203455645871538
36	0.203455645871545	0.203455645871540
37	0.203455645871541	0.203455645871541
38	0.203455645871545	0.203455645871542
39	0.203455645871541	0.203455645871542
40	0.203455645871544	0.203455645871542
41	0.203455645871542	0.203455645871543
42	0.203455645871544	0.203455645871543
43	0.203455645871542
44	0.203455645871544
45	0.203455645871542
46	0.203455645871544
47	0.203455645871542
48	0.203455645871544
49	0.203455645871542
50	0.203455645871544

Example 2 (increasing function)

Let $f: [0,8] \rightarrow [0,8]$ be defined as $f(x) = \frac{x^2+9}{10}$ with fixed point $q = 1.0000000000000000$. The comparison of convergence of the above iterative scheme is shown in the Table below.

n	CR iteration	Noor iteration	SP iteration	Ishikawa iteration
0	0.8000000000000000	0.8000000000000000	0.8000000000000000	0.8000000000000000
1	0.990509883403466	0.931873134968493	0.984457149643846	0.9640000000000000
2	0.999254290751118	0.967467541152655	0.997585405337407	0.9929296000000000

3	0.999928436979066	0.981930927764221	0.999478653195057	0.998590919055616
4	0.999992279006256	0.989031526654131	0.999861909338792	0.999718382362034
5	0.999999098786516	0.992932456751799	0.999957832431559	0.999943684403256
6	0.999999888628183	0.995244175790939	0.999985683103413	0.999988737197796
7	0.999999985623201	0.996691589860822	0.999994719300533	0.999997747452244
8	0.999999998078834	0.997636771298751	0.999997917449731	0.999999549490956
9	0.999999999735933	0.998274930507059	0.999999131863065	0.999999909898212
10	0.99999999962840	0.998717691080379	0.999999620724681	0.999999981979643
11	0.99999999994665	0.999031941101376	0.999999827481745	0.999999996395929
12	0.99999999999221	0.999259325753693	0.999999918721692	0.99999999279186
13	0.999999999999885	0.999426615209429	0.999999960503477	0.99999999855837
14	0.999999999999983	0.999551490501051	0.999999980271113	0.99999999971168
15	0.999999999999998	0.999645905062626	0.999999989898810	0.99999999994234
16	1.000000000000000	0.999645905062626	0.999999994711511	0.99999999998847
17	1.000000000000000	0.999773885492758	0.999999997174483	0.99999999999769
18	0.999817377166739	0.999999998462148	0.99999999999954
19	0.999851572569122	0.999999999148615	0.99999999999991
20	0.999878664753640	0.999999999521194	0.99999999999998
21	0.999900280050186	0.999999999726779	1.000000000000000
22	0.999917637269076	0.999999999841969	1.000000000000000
23	0.999931658647472	0.999999999907433
24	0.999943048211615	0.999999999945136
25	0.999952347808954	0.999999999967119
26	0.999959977628115	0.999999999980088
27	0.999966265807667	0.999999999987822
28	0.999971470281183	0.999999999992483
29	0.999975795038991	0.999999999995319
30	0.999979402334778	0.999999999997061
31	0.999982421919712	0.999999999998140
32	0.999984958078889	0.999999999998814

33	0.999987095029834	0.999999999999238
34	0.999988901090978	0.999999999999507
35	0.999990431919862	0.999999999999679
36	0.999991733043072	0.999999999999790
37	0.999992841843501	0.999999999999861
38	0.999993789129334	0.999999999999908
39	0.999994600378815	0.999999999999939
40	0.999995296732356	0.999999999999959
41	0.999995895786734	0.999999999999972
42	0.999996412233528	0.999999999999981
43	0.999996858374387	0.999999999999987
44	0.999997244538476	0.999999999999991
45	0.999997579421912	0.999999999999994
46	0.999997870364721	0.999999999999996
47	0.999998123577582	0.999999999999997
48	0.999998344328052	0.999999999999998
n	CR iteration	Noor iteration	SP iteration	Ishikawa iteration
49	0.999998537093986	0.999999999999999
50	0.999998705690304	1.000000000000000
51	0.999988533740370	1.000000000000000

CONCLUSION

We conclude that for decreasing function SP iterative scheme converges faster than Ishikawa and Noor Iterative schemes and CR iteration does not converge. For increasing function we conclude that for the choice $\alpha_n = \frac{1}{(1+n)^{\frac{1}{2}}}$, CR iterative scheme converges faster than SP, Noor and Ishikawa iterative schemes.

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